Lectures on
Polyhedral Topology

By

John R. Stallings

Tata Institute of Fundamental Research, Bombay
1967

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Introduction

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The recent efflorescence in the theory of polyhedral manifolds due to Smale's handle-theory, the differential obstruction theory of Munkres and Hirsch, the engulfing theorems, and the work of Zeeman, Bing and their students - all this has led to a wide gap between the modern theory and the old foundations typified by Reidemeister's Topologie der Polyeder and Whitehead's "Simplicial spaces, nuclei, and m-groups". This gap has been filled somewhat by various sets of notes, notably Zeeman's at I.H.E.S.; another interesting exposition is Glaser's at Rice University.

Well, here is my contribution to bridging the gap. These notes contain:

(1) The elementary theory of finite polyhedra in real vector spaces. The intention, not always executed, was to emphasize geometry, avoiding combinatorial theory where possible. Combinatorially, convex cells and bisections are preferred to simplexes and stellar or derived subdivisions. Still, some simplicial technique must be slogged through.

(2) A theory of "general position" (i.e., approximation of maps by ones whose singularities have specifically bounded dimensions), based on "non-degeneracy". The concept of n-manifold is generalized in the most natural way for general-position theory by that of $ND(n)$-space - polyhedron M such that every map from an n-dimensional polyhedron into M can be approximated by a non-degenerate map (one whose point-inverse are all finite). 2

(3) A theory of "regular neighbourhoods" in arbitrary polyhedra. Our regular neighbourhoods are all isotopic and equivalent to the star in a second-derived neighbourhoods are all isotopic and equivalent to the star in a second-derived subdivision (this is more or less the definition). Many applications are derived right after the elementary lemma that "locally collared implies collared". We then characterize regular neighbourhoods in terms of Whitehead's "collapsing", suitably modified for this presentation. The advantage of talking about regular neighbourhoods in arbitrary polyhedra becomes clear when we see exactly how they should behave at the boundaries of manifolds.

After a little about isotopy (especially the "cellular moves" of Zeeman), our description of the fundamental techniques in polyhedral topology is over. Perhaps the most basic topic omitted is the theory of block-bundles, microbundles and transversality.

(4) Finally, we apply our methods to the theory of handle - presentations of *PL*-manifolds à la Smalés theory for differential manifolds. This we describe sketchily; it is quite analogous to the differential case. There is one innovation. In order to get two handles which homotopically cancel to geometrically cancel, the "classical" way is to interpret the hypothesis in terms of the intersection number of attaching and transverse spheres, to reinterpret this geometrically, and then to embed a two-cell over which a sort of Whitney move can be made to eliminate a pair of intersection. Our method, although rather ad-hoc, is more direct, avoiding the algebraic complication of intersection numbers (es-
3 pecially unpleasant in the non-simply-connected case) as well as any worry that the two-cell might cause; of course, it amounts to the same thing really. This method is inspired by the engulfing theorem. [There are, by the way, at least two ways to use the engulfing theorem itself to prove this point].

We do not describe many applications of handle-theory; we do obtain Zeeman's codimension 3 unknotting theorem as a consequence.

This way of proving it is, unfortunately, more mundane than "sunny collapsing".

We omit entirely the engulfing theorems and their diverse applications. We have also left out all direct contact with differential topology.

* * *

Let me add a public word of thanks to the Tata Institute of Fundamental Research for giving me the opportunity to work on these lectures for three months that were luxuriously free of the worried, anxious students and administrative annoyances that are so enervating elsewhere. And many thanks to Shri Ananda Swarup for the essential task of helping write these notes.

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Bombay
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Chapter 1

Polyhedra

1.1 Definition of Polyhedra

Basic units out of which polyhedra can be constructed are convex hulls 4
of finite sets. A *polyhedron* (euclidean polyhedron) is a subset of some finite dimensional real vector space which is the union of finitely many such units. ("Infinite polyhedra" which are of interest in some topological situations will be discussed much later).

A *polyhedral map* $f : P \to Q$ is a function $f : P \to Q$ whose graph is a polyhedron. That is, suppose P and Q are subsets of vector spaces V and W respectively; the graph of f, denoted by $\Gamma(f)$, is the set

$$\Gamma(f) = \{(x, y) | x \in P,\ y = f(x) \in Q\}$$

which is contained in $V \times W$, which has an evident vector space structure. $\Gamma(f)$ is a polyhedron, if and only if (by definition), f is a polyhedral map. Constant functions, as well as identity function $P \to P$ are polyhedral maps.

The question whether the composition of polyhedral maps is polyhedral leads directly to the question whether the intersection of two polyhedra is a polyhedron. The answer is "Yes" in both cases. This could be proved directly, but we shall use a round about method which introduces uneful techniques.

It will be seen that polyhedra and polyhedral maps form a category. We are intersted in '*equivalences*' in this category, that is maps 5

$f : P \to Q$, which are polyhedral, one-to-one and onto. When do such equivalences exist? How can they be classified? Etc...

A finite dimensional real vector space V has a unique interesting topology, which can be described by any Euclidean metric on it. Polyhedra inherit a relative topology which make them compact metric spaces. Since polyhedral maps have compact graphs they are continuous. This provides us with an interesting relationship between polyhedra and topology. We may discuss topological matters about polyhedra - homology, homotopy, homeomorphy - and ask whether these influence the polyhedral category and its equivalences.

After this brief discussion of the space of the subject, we proceed to the development of the technique.

1.2 Convexity

$\mathbb{R}$ denotes the filed of real numbers, and V a finite dimensional vector space over.

Let $a, b \in V$. The *line segment* between a and b is denoted by $[a, b]$. It is defined thus:

$$[a, b] = \{ta + (1 - t)b | 0 \leq t \leq 1\}.$$

A set $C \subset V$ is called *convex* if $[a, b] \subset C$ whenever $a, b \in C$.

Clearly V itself is convex, and the intersection of any family of convex sets is again convex. Therefore every set $X \subset V$ is contained in a smallest convex set - namely the intersection of all convex sets containing X; this is called the *convex hull* of X, and is denoted by $K(X)$.

6 **Definition 1.2.1.** *A convex combination* of a subset X of V is a point of V which can be represented by a finite linear combination

$$\sum_{i=0}^{k} r_i x_i$$

where $x_i \in X$, $r_i \in \mathbb{R}$, $r_i \geq 0$ for all i, and $\sum_{i=0}^{k} r_i = 1$.

Proposition 1.2.2. *The convex hull $K(X)$ of X is equal to the set of convex combinations of X.*

Proof. Call the latter $\lambda(X)$. It will be shown first that $\lambda(X)$ is convex and contains X, hence $K(X) \subset \lambda(X)$.

It $x \in X$, then $1{\cdot}x$ is a convex combination of X, hence $X \supset \lambda(X)$. Let $\rho = \sum\limits_{i=0}^{k} r_i x_i$, $\sigma = \sum\limits_{j=0}^{\ell} s_j y_j$ be two points of $\lambda(X)$. A typical point of $[\rho, \sigma]$ is of the form $t\rho + (1-t)\sigma = \sum\limits_{i=0}^{k}(tr_i)x_i + \sum\limits_{j=0}^{\ell}((1-t)s_j)y_j$, where $0 \leq t \leq 1$. Since $\sum\limits_{i=0}^{k} tr_i + \sum\limits_{j=0}^{\ell}(1-t)s_j = t(\sum\limits_{i=0}^{k} r_i) + (1-t)(\sum\limits_{j=0}^{\ell} s_j) = t + (1-t) = 1$, and all the coefficients are ≥ 0, $t\rho + (1-t)\sigma$ is a convex combination of X. Hence $\lambda(X)$ is convex.

To show that $\lambda(X) \subset K(X)$ it must be shown that any convex set C containing X contains $\lambda(X)$. Let $\rho = r_1 x_1 + \cdots + r_n x_n$, $(x_i \in X, \sum r_i = 1)$ be a typical convex combination of $x_1, \ldots, x_n$. By induction on n it will be shown that any convex set C containing X contains ρ also. If $n = 1$,
$= x_i \in X \subset C$. If $n > 1$, then 7

$$\rho = r_1 x_1 + (1 - r_1)\left(\frac{r_2}{1 - r_1} x_2 + \cdots + \frac{r_n}{1 - r_1} x_n\right).$$

That is ρ is on the line segment between x_1 and $\frac{r_2}{1-r_1}x_2 + \cdots + \frac{r_n}{1-r_1}x_n$. By induction, the second point belongs to C, hence $\rho \in C$. Thus $\lambda(X) \subset C$. Therefore $\lambda(X) \subset K(X)$; and $\lambda(X) = K(X)$. □

Definition 1.2.3. A finite subset $\{x_0, \ldots, x_k\}$ of V is said to be *independent* (or *affinely independent*), if, for real numbers $r_0, \ldots, r_k$, the equations

$$r_0 x_0 + \cdots + r_k x_k = 0 \quad \text{and}$$
$$r_0 + \cdots + r_k = 0,$$

inply that

$$r_0 = \ldots = r_k = 0.$$

Ex. 1.2.4. The subset $\{x_0, \ldots, x_k\}$ of V is independent if and only if the subset $\{(x_0, 1), \ldots, (x_k, 1)\}$ of $V \times \mathbb{R}$ is linearly independent.

Ex. 1.2.5. The subset $\{x_0, \ldots, x_k\}$ of V is independent if and only if the subset $\{x_1 - x_0, \ldots, x_k - x_0\}$ of V is linearly independent.

Hence if $\{x_0, \ldots, x_k\} \subset V, x \in V$, then $\{x_0, \ldots, x_k\}$ is independent if and only if $\{x + x_0, \ldots, x + x_k\}$ is independent.

These two exercises show that the maximum number of independent points in V is $(\dim V + 1)$.

8 The convex hull of an independent set $\{x_0, \ldots, x_k\}$ is called a *closed k-simplex* with *vertices* $\{x_0, \ldots, x_k\}$ and is denoted by $[x_0, \ldots, x_k]$. The number k is called the *dimension* of the simplex.

The empty set $\emptyset$ is independent, its convex hull, also empty, is the unique (-1)-dimensional simplex. A set of only one point is independent; $[x] = \{x\}$ is a 0-dimensional simplex. A set of two distinct points is independent; the closed simplex with vertices $\{x, y\}$ coincides with the line segment $[x, y]$ between x and y.

Proposition 1.2.6. *If $\{x_0, \ldots, x_n\} \subset V$, then $\{x_0, \ldots, x_n\}$ is independent if and only if every point of $K\{x_0, \ldots, x_n\}$ is a unique convex combination of $\{x_0, \ldots, x_n\}$.*

Proof. Let $\{x_0, \ldots, x_n\}$ be independent. If $\rho : r_0x_0 + \cdots + r_nx_n = s_0x_0 + \cdots + s_nx_n$, with $\sum r_i = 1 = \sum s_i$, then $(r_0 - s_0)x_0 + \cdots + (r_n - s_n)x_n = 0$, and $(r_0 - s_0) + \cdots + (r_n - s_n) = 0$. Hence $(r_i - s_i) = 0$ for all i, and the expression for ρ is unique.

If $\{x_0, \ldots, x_n\}$ is not independent, then there are real numbers r_i, not all zero such that

$$r_0x_0 + \cdots + r_nx_n = 0 \quad \text{and}$$
$$r_0 + \cdots + r_n = 0.$$

Choose the ordering $\{x_0, \ldots, x_n\}$ so that there is a ℓ for which

$$r_i \geq 0 \quad \text{if} \quad i < \ell$$
$$r_i \leq 0 \quad \text{if} \quad i \geq \ell.$$

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Since not all r_i are zero, $r_0 + \cdots + r_{\ell-1} = (-r_\ell) + \cdots + (r_n) \neq 0$. Let this number be r. Then

$$\frac{r_0}{r}x_0 + \cdots + \frac{r_{\ell-1}}{r}x_{\ell-1} = \frac{-r_\ell}{r}x_\ell + \cdots + \frac{-r_n}{r}x_n.$$

But these are two distinct convex combinations of $\{x_1, \ldots, x_n\}$ which represent the same point, a contradiction. ◻

Proposition 1.2.7. *The convex hull $K(X)$ of X is equal to the union of all simplexes with vertices belonging to X.*

Proof. By 1.2.2, it is enough to show that a convex combination of X belongs to a simplex with vertices in X. Let $\rho = r_1x_1 + \cdots + r_nx_n$; $x_i \in X$, $\sum r_i = 1$, $r_i \geq 0$, be point of $K(X)$. It will be shown by induction on n that ρ belongs to a simplex with vertices in the set $\{x_1, \ldots, x_n\}$. If $n = 1$, then $\rho = x_1 \in [x_1]$. So let $n > 1$.

If $\{x_1, \ldots, x_n\}$ is independent, there is nothing to prove. If not, there are $s_1, \ldots, s_n$, not all zero, such that $s_1x_1 + \cdots + s_nx_n = 0$ and $s_1 + \cdots + s_n = 0$. When $s_i = 0$, define $\dfrac{r_i}{s_i} = \infty$; then it can be supposed that $x_1, \ldots, x_n$ is arranged such that

$$\left|\frac{r_1}{s_1}\right| \geq \left|\frac{r_2}{s_2}\right| \geq \ldots \geq \left|\frac{r_n}{s_n}\right|.$$

Then $s_n \neq 0$. Hence $x_n = -\dfrac{1}{s_n}(s_1x_1 + \cdots + s_{n-1}x_{n-1})$.

Therefore 10

$$\rho = \left(r_1 - s_1\frac{r_n}{s_n}\right)x_1 + \left(r_2 - s_2\frac{r_n}{s_n}\right)x_2$$
$$+ \cdots + \left(r_{n-1} - s_{n-1}\frac{r_n}{s_n}\right)x_{n-1}.$$

Since for all $i < n$, $\left|\frac{r_i}{s_i}\right| \geq \left|\frac{r_n}{s_n}\right|$, and since $-\frac{s_1}{s_n} - \ldots - \frac{s_{n-1}}{s_n} = 1$, this expresses ρ as a convex combination of $\{x_1, \ldots, x_{n-1}\}$. By inductive hypothesis, ρ is contained in a simplex with vertices in $\{x_1, \ldots, x_{n-1}\}$. ◻

The following propositions about independent sets will be useful later (See L.S. Pontryagin “Foundations of combinatorial Topology”, Graylock Press, Rochester, N.Y., pages 1-9 for complete proofs).

Let $\dim V = m$, and δ be a euclidean metric on V. First, propositon 1.2.4 can be reformulated as follows:

Ex. 1.2.8. Let $\{e_1, \ldots, e_m\}$ be a basis for V, and $\{x_0, \ldots, x_n\}$ a subset of V. Let $x_i = a_i^1 e_1 + \cdots + a_i^m e_m$; $0 \leq i \leq n$. Then the subset $\{x_0, \ldots, x_n\}$ is independent if and only if the matrix

$$\begin{bmatrix} 1 \; a_0^1 & a_0^2 \ldots a_0^m \\ 1 \; a_1^1 & a_1^2 \ldots a_1^m \\ \ldots & \ldots \\ 1 \; a_n^1 & a_n^2 \ldots a_n^m \end{bmatrix}$$

has rank $(n + 1)$.

11 **Proposition 1.2.9.** *Let $\{x_0, \ldots, x_n\}$ be a subset of V, $n \leq m$. Given any $(n + 1)$ real numbers $\epsilon_i > 0$, $0 \leq i \leq n$, $\exists$ points $y_i \in V$, such that $\delta(x_i, y_i) < \epsilon_i$, and the set $\{y_0, \ldots, y_n\}$ is independent.*

Sketch of the proof: Choose a set $\{u_0, \ldots, u_n\}$ of $(n + 1)$ independent points and consider the sets $Z(t) = \{t\, u_0 + (1 - t)x_0, \ldots, tu_n + (1 - t)x_n\}$, $0 \leq t \leq 1$. Let $N(Z(t))$ denote the matrix corresponding to the set $Z(t)$ as given in 1.2.8. (the points being taken in the particular order). $Z(1) = \{u_0, \ldots, u_n\}$, hence some matrix of $(n + 1)$-columns of $N(Z(1))$ has nonzero determinant. Let $D(t)$ denote the determinant of the corresponding matrix in $N(Z(t))$. $D(t)$ is a polynomial in t, and does not vanish identically. Hence there are numbers as near 0 as we like such that $D(s)$ does not vanish. This means that $N(Z(s))$ is independent, and if s in near 0, $Z(s)_i$ will be near x_i. $\square$

Hence in any arbitrary neighbourhood of a point of V, there are $(m+1)$ independent points.

The above proof is reproduced from Pontryagm’s book. The next propositions are also proved by considering suitable determinants (see the book of Pontryagin mentioned above).

Ex. 1.2.10. If the subset $\{x_0, \ldots, x_n\}$ of V is independent, then there exists a number $\eta > 0$, such that any subset $\{y_0, \ldots, y_n\}$ of V with $\delta(x_i, y_i) < \eta$ for all i, is again independent.

Ex. 1.2.11. A subset $X = \{x_0, \ldots, x_n\}$ of V is said to be in *general position*, if every subset of X containing $m + 1$ points is independent (where $m = \dim V$).

Ex. 1.2.12. Given any subset $X = \{x_0, \ldots, x_n\}$ of V and $(n + 1)$-numbers 12
$\epsilon_i > 0$, $0 \leq i \leq n$, there exists points y_i, $0 \leq i \leq n$ with $\delta(x_i, y_i) < \epsilon_i$, and such that the subset $Y = \{y_0, \ldots, y_n\}$ of V is in general position.

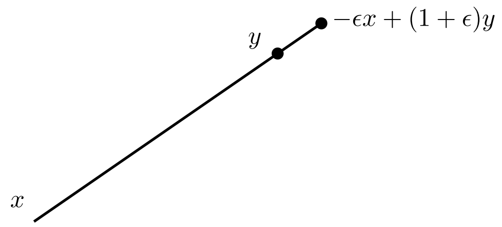
Hint: Use 1.2.9, 1.2.10 and induction.

1.3 Openconvex sets

Definition 1.3.1. A subset A of V is said to be an *open convex set* if

(1) A is convex

(2) for every $x, y \in A$, there exists 0, such that $-\epsilon x + (1 + \epsilon)y \in A$, ($\epsilon = \epsilon(x, y)$ depending on x, y).



In otherwords the line segment jointing x and y can be prolonged a little in A.

Clearly the empty set and any set consisting of one point are open convex sets. So open convex sets in V need not necessarily be open in the topology of V.

Clearly the intersection of finitely many open convex sets is again an open convex set.

Definition 1.3.2. Let $\{x_1, \ldots, x_n\} \subset V$.

13 An *open convex combination* of $\{x_1, \ldots, x_n\}$ is a convex combination $r_1 x_1 + \cdots + r_n x_n$ such that every coefficient $r_i > 0$. The set of all points represented by such open convex combinations is denoted by $0(x_1, \ldots, x_n)$.

It is easily seen that $0(x_1, \ldots, x_n)$ is an open convex set.

Definition 1.3.3. If $\{x_0, \ldots, x_k\}$ is independent, then $0(x_0, \ldots, x_k)$ is called an open k-simplex with vertices $\{x_0, \ldots, x_k\}$. The number k is called the dimension of the simplex $0(x_0, \ldots, x_k)$. If $\{i_0, \ldots, i_s\} \subset \{0, \ldots, n\}$, then the open simplex $0(x_{i_0}, \ldots, x_{i_s})$ is called a *s-face* (or a face) of $0(x_0, \ldots, x_k)$. If $s < k$, then, it is called a *proper face*.

Clearly, the closed simplex $[x_0, \ldots, x_k]$ is the disjoint union of $0(x_0, \ldots, x_k)$ and all its proper faces.

We give another class of examples of open convex sets below which will be used to construct other types of open convex sets.

Definition 1.3.4. A *linear manifold* in V is a subset M of V such that whenever $x, y \in M$ and $r \in \mathbb{R}$ then $rx + (1 - r)y \in M$.

Linear manifolds in V are precisely the translates of subspaces of V; that is, if V' is a subspace of V, and $z \in V$, then the set $z + V' = \{z + z' | z' \in V'\}$ is a linear manifold in V, and every linear manifold in V is of this form. Moreover, given a linear manifold M the subspace V_M of V of which M is a translate is unique, namely,

$$V_M = \{z - y | z \in M, y \in M\} = \{z - z' | z \in M, z' \text{ a fixed element of } M\}.$$

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Thus the *dimension* of a linear manifold can be easily defined, and is equal to one less than the cardinality of any maximal independent subset of M (see 1.2.5). A linear manifold of dimension 1, we will call a *line*. If L is a line, $a, b \in L$, $a \neq b$, then every other point on L is of the form $ta + (1 - t)b$, $t \in \mathbb{R}$. If M is a linear manifold in V and $\dim M = (\dim V - 1)$, then we call M a *hyperplane* in V.

Definition 1.3.5. Let V and W be real vector spaces. A function $\varphi : V \to W$ is said to be a linear map, if for every $t \in \mathbb{R}$ and every $x, y \in V$,

$$\varphi(tx + (1 - t)y) = t\varphi(x) + (1 - t)\varphi(y).$$

Alternatively, one can characterize a linear map as being the sum of a vector space homomorphism and a constant.

Ex. 1.3.6. In definition 1.3.5, it is enough to assume the $\varphi(tx+(1-t)y) = t\varphi(x) + (1 - t)\varphi(y)$ for $0 \leq t \leq 1$.

If A is a convex set in V and $\varphi : A \to W$, (W a real vector space) is a map such that, for $x, y \in A$, $0 \leq t \leq 1$

$$\varphi(tx + (1 - t)y) = t\varphi(x) + (1 - t)\varphi(y),$$

then also we call φ *linear*. It is easy to see that φ is the restriction to A of a linear map of V (which is uniquely defined on the linear manifold spanned by A).

Ex. 1.3.7. Let A, V, W be as above and $\varphi : A \to W$ a map. Show that φ is linear if and only if the graph of φ is convex. (graph of φ is the subset of $V \times W$ consisting of (x, y), $x \in A$, $y = \varphi(x)$).

Ex. 1.3.8. The images and preimages of convex sets under a linear map **15**
(resp. open convex sets) are convex sets (resp. open convex sets). The images and preimages of linear manifolds under a linear map are again linear manifolds.

A hyperplane P in V for instance is the preimage of 0 under a linear map from V to $\mathbb{R}$. Thus with respect to some basis of V, P is given by an equation of the form $\sum \ell_i x_i = d$, where x_i are co-ordinates with respect to a basis of V and ℓ_i, $d \in \mathbb{R}$ not all the ℓ_i's being zero. Hence $V - P$ consists of two connected components ($\sum \ell_i x_i > d$ and $\sum \ell_i x_i < d$), which we will call the *half-spaces* of V determined by P. A half space of V is another example of an open convex set.

Definition 1.3.9. A *bisection* of a vector space V consists of a triple $(P; H^+, H^-)$ consisting of a hyperplane P in V and the two half spaces H^+ and H^- determined by P.

These will be used in the next few section. A few more remarks: Let the dimension of $V = m$ and V' be a $(m-k)$-dimensional subspace of V. Then extending a basis of V' to a basis of V we can express V' as the intersection of $(k-1)$ subspace of V of dimension $(m-1)$. Thus any linear manifold can be expressed as the intersection of finite set (non unique) of hyperplanes. Also we can talk of hyperplanes, linear submanifolds etc. of a linear manifold M in V. These could for example be taken as the translates of such from the corresponding subspace of V or we can consider them as intersections of hyperplanes and linear
16 manifolds in V with M. Both are equivalent. Next, the topology on V is taken to be topology induced by any Euclidean metric on V. The topology on subspaces of V inherited from V is the same as the unique topology defined by Euclidean metric on V. The topology on subspaces of V inherited from V is the same as the unique topology defined by Euclidean metrics on them. And for a linear manifold M we can either take the topology on M induced from V or from subspace of V of which it is a translate. Again both are the same. We will use these hereafter without more ado.

1.4 The calculus of boundaries

Definition 1.4.1. Let A be an open convex set in V. A point $x \in V - A$ is called a *boundary point* of A, if there exists a point $a \in A$ such that $O(x, a) \in A$. The set of all boundary points of A is called the *boundary* of A and is denoted by ∂A.

A number of propositions will now be presented as exercises, and sometimes hints are given in the form of diagrams. In each given context a real vector space is involved even when it is not explicitly mentioned, and the sets we are considering are understood to be subsets of that vector space.

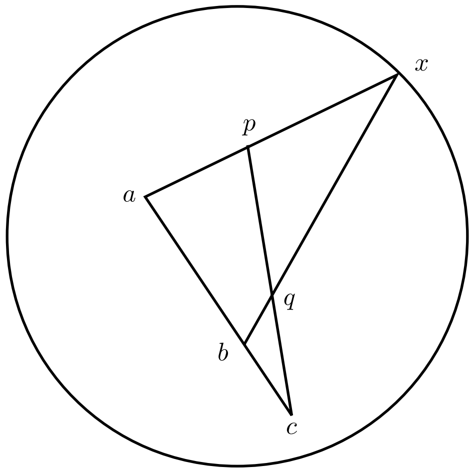
Ex. 1.4.2. A linear manifold has empty boundary. Conversely, if an open convex set A has empty boundary, then A is a linear manifold.

Remark. This uses the completeness of real numbers.

Ex. 1.4.3. If $(P; H^+, H^-)$ is a bisection of V, then $\partial H^+ = \partial H^- = P$ and $\partial P = \emptyset$.

Proposition 1.4.4. *If A is an open convex set and $x \in \partial A$, then for all $b \in A$, $0(x, b) \subset A$.*

Proof. Based on this picture: There is 'a' such that $0(x, a) \subset A$. Extend **17**
a, b to a point $c \in A$. For any $q \in 0(x, b)$, there exists a $p \in 0(x, a)$ such that $q \in 0(c, p)$. Since c, $p \in A$, $q \in A$. Hence $0(x, b) \subset A$.



□

Ex. 1.4.5. Let $\varphi : V \to W$ be a linear map, and let B be an open convex set in W. Then $\partial(\varphi^{-1}(B))(\varphi^{-1}(\partial B))$. If φ is onto then equality holds.

Definition 1.4.6. The *closure* of an open convex set A is defined to be $A \cup \partial A$; it is denoted by $\overline{A}$.

Ex. 1.4.7. If $A \subset B$, then $\overline{A} \subset \overline{B}$.

Proposition 1.4.8. *If $a, b \in A$ and $a \neq b$, where A is an open convex set, then there is at most one $x \in \partial A$ such that $b \in 0(a, x)$.*

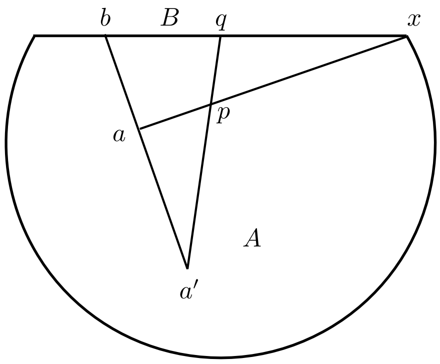
Proof. If $b \in 0(a, x)$ and $b \in 0(a, y)$; $x, y \in \partial A$, $x \neq y$, then $0(a, x)$ and $0(a, y)$ lie on the same line, the line through a and b and both, are on the same side of a as b. Either x or y must be closer to a i.e. either $x \in 0(a, y)$ or $y \in 0(a, x)$. If $x \in 0(a, y)$, then $x \in A$, but $A \cap \partial A = \emptyset$. Similarly $y \in 0(a, x)$ is also impossible. □

Proposition 1.4.9. *Let $\{x_0, \dots, x_n\}$ be an independent set whose convex hull is contained in ∂A, where A is an open convex set. Let $a \in A$. Then $\{x_0, \dots, x_n, a\}$ is independent.*

18 *Proof.* 1.4.8 shows that each point of $K\{x_0, \dots, x_n, a\}$ can be written as a unique convex combination. Hence by 1.2.6 $\{x_0, \dots, x_n, a\}$ is independent. □

Proposition 1.4.10. *Let A and B be open convex sets. If $B \subset \partial A$, then $\partial B \subset \partial A$.*

Proof. Based on this picture:



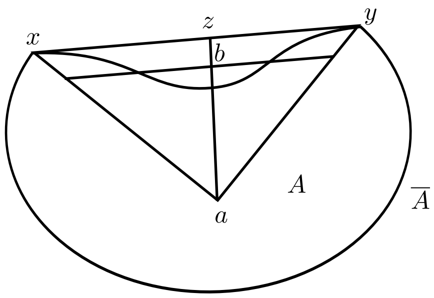
The case A or B is empty is trivial. Otherwise, let $x \in \partial B$, $b \in B$, $a \in A$; extend the segment $[b, a]$ to $a' \in A$. Let $p \in 0(x, a)$; then $q \in 0(x, b)$ can be found such that $p \in 0(q, a')$. Since $q \in 0(x, b) \subset B \subset \partial A$, it follows that $0(q, a') \subset A$, therefore $p \in A$. Hence $0(x, a) \subset A$; obviously x does not belong to A and so $x \in \partial A$. □

Definition 1.4.11. If A and B are open convex sets, define $A < B$ to mean $A \subset \partial B$.

1.4.10 implies that $<$ is transitive.

Ex. 1.4.12. If A is an open convex set, then $\overline{A}$ is convex.

Hint:



□

Proposition 1.4.13. *If A and B are open convex sets with $A \cap B \neq \emptyset$, then $\partial(A \cap B)$ is the disjoint union of $\partial A \cap B$, $A \cap \partial B$ and $\partial A \cap \partial B$.* 19

Proof. These three sets are disjoint, since $A \cap \partial A = B \cap \partial B \neq \emptyset$. Let $\mathscr{C} \in A \cap B$ and $x \in \partial(A \cap B)$; since $x \in V - (A \cap B) = (V - A) \cup (V - B)$, x either (1) belongs to $V - A$ and to $V - B$ or (2) belongs to $V - A$ and to B or (3) belongs to A and to $V - B$. Since $0(x, c) \subset A \cap B$, in case (1) $x \in \partial A \cap \partial B$, in case (2) $x \in \partial A \cap B$ and in case (3) $x \in A \cap \partial B$. The converse is similarly easy. □

Another way of stating 1.4.13 is to say that $\overline{A} \cap \overline{B} = \overline{A \cap B}$, when $A \cap B \neq \emptyset$.

Proposition 1.4.14. *If A and B are open convex sets and $A \subset \overline{B}$ and $A \cap B \neq \emptyset$, then $A \subset B$.*

Proof. Let $c \in A \cap B$, and $a \in A$. The line from 'c' to 'a' may be prolonged a little bit to $a' \in A$. Since $a' \in \overline{B}$, it follows that $0(a', c) \subset B$, but $a \in 0(a', c)$. Hence $A \subset B$. □

Proposition 1.4.15. *If $\overline{A} = \overline{B}$, where A and B are open convex sets, then $A = B$.*

Proof. If $A \cap B = \emptyset$, since $A \cup \partial A = B \cup \partial B$, we have $A \subset \partial B$ and $B \subset \partial A$. By 1.4.10 we have $A \subset \partial A$ and $B \subset \partial B$. But $A \cap \partial A = \emptyset = B \cap \partial B$. Hence $A \cap B = \emptyset$ is impossible except for the empty case. Then by 1.4.14, $A \subset B$ and $B \subset A$. Therefore $A = B$. □

Proposition 1.4.16. *Let $\overline{0}(x_1, \ldots, x_n)$ denote the closure of $0(x_1, \ldots, x_n)$. Then $\overline{0}(x_1, \ldots, x_n) = K\{x_1, \ldots, x_n\}$.*

20 *Proof.* First, $K\{x_1, \ldots, x_n\} \subset \overline{0}(x_1, \ldots, x_n)$. For let $y \in K\{x_1, \ldots, x_n\}$; then y is a convex linear combination $r_1x_1 + \cdots + r_nx_n$. Let $z = \dfrac{1}{n}(x_1 + \cdots + x_n) \in 0(x_1, \ldots, x_n)$. Then every point on the line segment $0(y, z)$ is obviously expressed as an open convex combination of $x_1, \ldots, x_n$; hence $0(y, z) \subset 0(x_1, \ldots, x_n)$, and so $y \in \overline{0}(x_1, \ldots, x_n)$.

Conversely, let $y \in \overline{0}(x_1, \ldots, x_n)$. If $y \in 0(x_1, \ldots, x_n)$, clearly $y \in K\{x_1, \ldots, x_n\}$. Suppose $y \in \partial 0(x_1, \ldots, x_n)$; let $z = \dfrac{1}{n}(x_1 + \cdots + x_n)$ as above. On the line segment $0(y, z)$, pick a sequence a_i of points tending to y. Now, $a_i \in 0(y, z) \subset 0(x_1, \ldots, x_n) \subset K\{x_i, \ldots, x_n\}$. Let $a_i = r_{i_1}x_1 + \cdots + r_{i_n}x_n$. r_{i_j} are bounded by 1. By going to subsequences if necessary we can assume that the sequences $\{r_{i_j}\}$ converge for all j, say to r_j. Then $\sum r_j = 1$, $r_j \geq 0$, and $\sum r_{i_j}x_j$ converge to $\sum r_jx_j \in K\{x_1, \ldots, x_n\}$. But $\sum r_{i_j}x_j$ also converge to y. Hence $y = \sum r_jx_j$ and $y \in K\{x_1, \ldots, x_n\}$. □

Definition 1.4.17. An open convex set A is said to be bounded, if for every line L in V, there are points $x, y \in L$, such that $A \cap L \subset [x, y]$.

Since in any case $A \cap L$ is an open convex set, either $A \cap L$ is empty, or $A \cap L$ consists of a single point, or $A \cap L$ is an open interval, possibly infinite on L. The boundedness of A then implies that if $A \cap L$ contains
21 at least two points, there are $x, y \in L$ such that $A \cap L = 0(x, y)$.

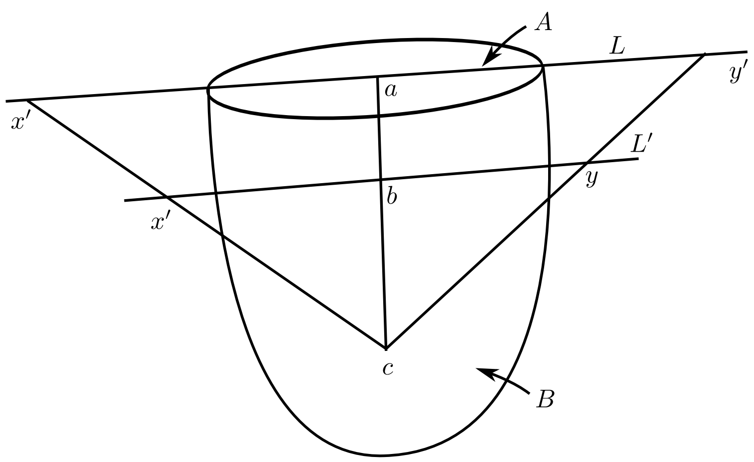
Proposition 1.4.18. *If A is a bounded open convex set containing at least two points, then $\overline{A} = K(\partial A)$.*

Proof. Since $\partial A \subset \overline{A}$ and $\overline{A}$ is convex, it is always true that $K(\partial A) \subset \overline{A}$. Clearly $\partial A \subset K(\partial A)$. It remains only to show that $A \subset K(\partial A)$. Let $a \in A$. Let L be aline through 'a' and another point $b \in A$ (such another point exists by hypothesis). Since A is bounded, and $\{a, b\} \in L \cap A$, it follows that $A \cap L = 0(x, y)$ for some $x, y \in L$. Clearly $x, y \in \partial A$, and $a \in [x, y] \subset K(\partial A)$. □

Remark. With the hypothesis of 1.4.18, we have $\overline{A} = \bigcup_y [a, y]$, '$a$' a fixed point of A and $y \in \partial A$ and $A = \bigcup_y 0(a, y) \cup \{a\}$.

Ex. 1.4.19. If A and B are open convex sets, and $A < B$, and B is bounded, then A is bounded.

Hint:



□

Ex. 1.4.20. Let A be an open convex set in V, and B be an open convex set in W. Then (1) $A \times B$ is an open convex set in $V \times W$; (2) $\partial(A \times B)$ is the disjoint union of $\partial A \times B$, $A \times \partial B$ and $\partial A \times \partial B$; (3) $A \times B$ is bounded if and only if A and B are, provided $A \neq \emptyset$, $B \neq \emptyset$. 22

The following two exercises are some what difficult in the sense they use compactness of the sphere, continuity of certain functions etc.

Ex. 1.4.21. The closure of A defined above (1.4.6) coincides with the topological closure of A in V.

Ex. 1.4.22. An open convex set which is bounded in the sense of some Euclidean metrix is bounded in the above sense, and conversely.

1.5 Convex cells

Definition 1.5.1. An *open convex cell* is defined to be a finite intersection of hyperplanes and half spaces, which as an open convex set is bounded.

Clearly the intersection of two open convex cells is an open convex cell, and the product of two open convex cells is an open convex cell.

With respect to a coordinate system in the vector space in which it is defined, an open convex cell is given by a finite system of linear inequalities. If A is an open convex cell, by taking the intersection of all the hyperplanes used in defining A, we can write $A = P \cap H_1 \cap \ldots \cap H_\ell$,
23 where P is a linear manifold and H_i are half spaces. Since H_i are open in the ambient vector space A is open in P. Let $A = P' \cap H'_1 \cap \ldots \cap H'_{\ell'}$ be another such representation of A. If A is nonempty, then $P = P'$. For $A \subset P \cap P'$ and if $P \neq P'$, $P \cap P'$ is of lower dimension than P, hence A cannot be open in P. Thus $P' = P$; though the H_i's and H'_j's may differ. Hence P can be described as the unique linear manifold which contains A as an open subset. We define the *dimension* of the open convex cell A to be the dimension of the above linear manifold P. If $A = \emptyset$, we define the dimension of A to be -1.

If A is an open convex cell, we will call $\overline{A}$ a *closed convex cell*. The *boundary* of a closed convex cell is defined to be the same as the boundary of the open convex cell of which it is the closure. This is well defined, since $\overline{A} = \overline{B}$ implies $A = B$, when A and B are open convex sets (1.4.15). The *dimension* of $\overline{A}$ is defined to be the same as the dimension of A.

Using 1.2.9 and 1.2.10, it is easily seen that the dimension of A is one less than the cardinality of maximal independent set contained in A or $\overline{A}$. Similar remark applies for $\overline{A}$ also. Actually, using this description we can extend the definition of dimension to arbitrary convex sets.

Ex. 1.5.2. If A is an open convex cell of dimension K, and $A_1, \ldots, A_n$ are open convex cells of dimension $< K$, then $A \not\subset A_1 \cup \ldots \cup A_n$.

Proposition 1.5.3. *An open k-simplex is an open convex cell of dimension k. A closed k-simplex is a closed convex cell of dimension k.* **24**

Proof. It is enough to prove for the open k-simplex. Let the open k-simplex be $0(x_0, \ldots, x_k) = A$ in the vector space V. The the unique linear manifold P containing A is the set of points $r_0 x_0 + \cdots + r_k x_k$, where $r_0 + \cdots + r_k = 1$, $r_i \in \mathbb{R}$. Define $\varphi_i(r_0 x_0 + \cdots + r_k x_k) = r_i$; φ_i is a linear map from P to $\mathbb{R}$. Then $H_i = \varphi_i^{-1}(0, \infty)$ is a half space relative to the hyperplane P and $0(x_0, \ldots, x_k) = H_0 \cap \ldots \cap H_k$. By extending H_i to half spaces H_i' in V suitably, $0(x_0, \ldots, x_k) = P \cap H_0' \cap \ldots \cap H_k'$. Boundedness of A, and that $\dim A = k$ are clear. ◻

Proposition 1.5.4. *Let A be a nonempty open cell. Then there is a finite set $\mathscr{P} = \{A_1, \ldots, A_k\}$ whose elements are open convex cells, such that*

(a) $\overline{A} = \bigcup_{1 \le i \le k} A_i$.

(b) *$A_i \cap A_j = \emptyset$ if $i \neq j$*

(c) *A is one of the cells A_i*

(d) *The boundary of each element of $\mathscr{P}$ is union of elements of $\mathscr{P}$. (Of course the empty set is also taken as such a union).*

Proof. Let $A = P \cap H_1 \cap \ldots \cap H_n$, where P is a linear manifold and H_i are half spaces with boundary hyperplanes P_i. Let $\mathscr{P}$ be the set whose elements are nonempty sets of the following sort:

Let

$$\{1, \ldots, n\} = \{j_1, \ldots, j_q\} \cup \{k_1, \ldots, k_{n-q}\}.$$

Then if it is not empty the set $P \cap H_{j_1} \cap \ldots \cap H_{j_q} \cap P_{k_1} \cap \ldots \cap P_{k_{(n-q)}}$ is an element of P. The properties of $\mathscr{P}$ follow from 1.4.13. 25

If the above, the union of elements of $\mathscr{P}$ excluding A constitute the boundary ∂A of A. Then using 1.4.9, and the remarks preceding 1.5.2, we have, if $A_i \in \mathscr{P}$, $A_i \neq A$, then $\dim A_i < \dim A$. We have seen that if A is a bounded open convex set of dim ≥ 1, then $\overline{A} = K(\partial A)$. Hence by an obvious induction, we have □

Proposition 1.5.5. *A closed convex cell is the convex hull of a finite set of points.*

A partial converse of 1.5.5, is trivial:

1.5.6 The convex hull of a finite set is a finite union of open (closed) convex cells.

The converse of 1.5.5 is also true.

Ex. 1.5.7. The convex hull of a finite set is a closed convex cell.

Hint: Let $\{x_1, \ldots, x_n\}$ be a finite set in vector space V. By 1.4.16 $K\{x_1, \ldots, x_n\} = \overline{0}(x_1, \ldots, x_n)$. It is enough to show that $0(x_1, \ldots, x_n)$ is an open convex cell. Let M be the linear manifold generated by $\{x_1, \ldots, x_n\}$. Let $\dim M = k$. Write $A = 0(x_1, \ldots, x_n)$, $\overline{A} = K\{x_1, \ldots, x_n\}$. A is open in M. To prove the proposition it is enough to show that A is the inter section of half spaces in M.

Step 1. $\overline{A}$ and ∂A are both union of open (hence closed) simplexes with vertices in $\{x_1, \ldots, x_n\}$. The assertion for $\overline{A}$ follows from 1.2.7.

26 **Step 2.** If B is a $(k-1)$-simplex in ∂A and N is the hyperplane in M defined by B, then A cannot have points in both the half spaces defined by N in M.

Step 3. It is enough to show that each point of ∂A belongs to a closed $(k-1)$-simplex with vertices in $\{x_1, \ldots, x_n\}$.

Step 4. Each point $x \in \partial A$ is contained in a closed $(k-1)$-simplex with verticer in $\{x_1, \ldots, x_n\}$. To prove this let $C_1, \ldots, C_p$ be the closed simplexes contained in ∂A with vertices in $x_1, \ldots, x_n$ which contain x, and $D_1, \ldots, D_q (\subset \partial A)$ which do not contain x. By Step (1) $\bigcup\limits_i C_i \bigcup\limits_j D_j = \partial A$. Consider any point $a \in A$ and a point $b \in 0(a, x)$. Let C'_i (resp. D'_j) denote the closed simplex whose vertices are those of C_i (resp. D_j) and 'a'. By the remark following 1.4.18. $\bigcup\limits_i C'_i \bigcup\limits_j D'_j = \overline{A}$. Show that $\bigcup\limits_i C'_i$ is a neighbourhood of b. If $\dim C_i < k-1$ for all i, then $\dim C'_i \leq k-1$ for all i. Use 1.5.2 to show that in this case $\bigcup C'_i$ cannot be a neighbourhood of b.

Since the linear image of convex hull of a finite set is also the convex hull of finite set, 1.5.7, immediately gives that the linear image of a closed convex cell is a closed convex cell. If A is an open convex cell in V and φ a linear map from V to W, then $\varphi(\overline{A}) = \overline{\varphi(A)}$, by 1.4.16, hence by 1.4.15 $\varphi(A)$ is an open convex cell. Therefore

Proposition 1.5.8. *The linear image of an open (resp. closed) convex cell is an open (resp. closed) convex cell.*

1.6 Presentations of polyhedra

27

If $\mathscr{P}$ is a set of sets and A is set, we shall write

$$A \vee \mathscr{P}$$

when A is a union of elements of $\mathscr{P}$. For example (d) of 1.5.4 can be expressed as "If $A \in \mathscr{P}$, then $\partial A \vee \mathscr{P}$". We make the obvious convention, when $\emptyset$ is the empty set, that $\phi \vee \mathscr{P}$ no matter what $\mathscr{P}$ is.

Definition 1.6.1. A *polyhedral presentation* is a finite set $\mathscr{P}$ whose elements are open convex cells, such that $A \in \mathscr{P}$ implies $\partial A \vee \mathscr{P}$.

Definition 1.6.2. A *regular presentation* is a polyhedral presentation $\mathscr{P}$ such that any two distinct elements are disjoint, that is, $A \in \mathscr{P}$, $B \in \mathscr{P}$, $A \neq B$ implies $A \cap B = \emptyset$.

Ex. The $\mathscr{P}$ of proposition 1.5.4 is a regular presentation.

Definition 1.6.3. A *simplicial presentation* is a regular presentation whose elements are simplicies and such that if $A \in \mathscr{P}$, then every fact of A also belongs to $\mathscr{P}$.

If $\mathscr{Q} \subset \mathscr{P}$ are polyhedral presentations, we call $\mathscr{Q}$ a *subpresentation* of $\mathscr{P}$. If $\mathscr{P}$ is regular (resp. simplicial) then $\mathscr{Q}$ is necessarily regular (resp. simplicial). The points of the O-cells of a simplicial presentation will be called the *vertices* of the simplicial presentation. The *dimension* of a polyhedral presentation $\mathscr{P}$ is defined to be the maximum of the dimensions of the open cells of $\mathscr{P}$.

28 **Definition 1.6.4.** If $\mathscr{P}$ is a polyhedral presentation $|\mathscr{P}|$ will be used to denote the union of all elements of $\mathscr{P}$. We say that $\mathscr{P}$ *is a presentation* of $|\mathscr{P}|$ or that $|\mathscr{P}|$ *has a presentation* $\mathscr{P}$.

Recall that in 1.1, we have defined a polyhedron as a subset of a real vector space, which is a finite union of convex hulls of finite sets. It is clear consequence of 1.5.4, 1.5.5 and 1.5.6 that

Proposition 1.6.5. *Every polyhedron has a polyhedral presentation. If $\mathscr{P}$ is a polyhedral presentation, then $|\mathscr{P}|$ is a polyhedron.*

Thus, if we define a polyhedron as a subset of a real vector space which has a polyhedral presentation, then this definition coincides with the earlier definition.

Proposition 1.6.6. *The union or intersection of a finite number of polyhedra is again a polyhedron.*

Proof. It is enough to prove for two polyhedra say P and Q. Let $\mathscr{P}$ and $\mathscr{Q}$ be polyhedral presentations of P and Q respectively. Then $\mathscr{P} \cup \mathscr{Q}$ is a polyhedral presentation of $P \cup Q$; hence $P \cup Q$ is a polyhedron. To prove that $P \cap Q$ is a polyhedron, consider the set $\mathscr{R}$ consisting of all nonempty sets of the form $A \cap B$, for $A \in \mathscr{P}$ and $B \in \mathscr{Q}$. It follows from 1.4.13 that $\mathscr{R}$ is a polyhedral presentation. Clearly $|\mathscr{R}| = P \cap Q$. Hence by 1.6.5 $P \cap Q$ is a polyhedron. $\square$

If $X \subset Y$ are polyhedra, we will call X a *subpolyhedron* of Y. Thus in 1.6.6, $P \cap Q$ is a subpolyhedron of both P and Q.

1.6.7 If $\mathscr{P}$ and Q are two polyhedral presentations consider the sets of the form $A \times B$, $A \in \mathscr{P}$, $B \in Q$. Clearly $A \times B$ is an open convex **29**
cell, and by 1.4.20 $\partial(A \times B)$ is the disjoint union of $\partial A \times B$, $A \times \partial B$ and $\partial A \times \partial B$. Thus the set of cells of the form $A \times B$, $A \in \mathscr{P}$, $B \in Q$ is a polyhedral presentation, regular if both $\mathscr{P}$ and Q are. This we will denote by $\mathscr{P} \times Q$. As above, we have, as a consequence that $P \times Q$ is a polyhedron, with presentation $\mathscr{P} \times Q$.

Ex. 1.6.8. The linear image of a polyhedron is a polyhedron (follows from the definition of polyhedron and the definition of linear map).

Recall that we have defined a polyhedral map between two polyhedra as a map whose graph is a polyhedron.

Proposition 1.6.9. *The composition of two polyhedral maps is a polyhedral map.*

Proof. Let X, Y and Z be three polyhedra in the vector spaces U, V and W respectively, and let $f : X \to Y$, $g : Y \to Z$ be polyhedral maps. Then $\Gamma(f) \subset U \times V$ and $\Gamma(g) \subset V \times W$ are polyhedra. By 1.6.7, $\Gamma(f) \times Z$ and $X \times \Gamma(g)$ are also polyhedra in $U \times V \times W$. By 1.6.6, $(\Gamma(f) \times Z) \cap (X \times \Gamma(g))$ is a polyhedron. This intersection is the set

$$S : \{(x, y, z) \mid x \in X,\ y = f(x),\ z = g(y)\}$$

in $U \times V \times W$. By 1.6.8 the projection of $U \times V \times W$ to $U \times W$ takes S into a polyhedron, which is none other than the graph of the map $g \circ f : X \to Z$. Hence $g \circ f$ is polyhedral. □

If a polyhedral map $f : P \to Q$, is one-to-one and onto we term it a *polyhedral equivalence*.

Ex. 1.6.10. If, $f : P \to Q$ is a polyhedral map, then the map $g : P \to \Gamma(f)$ defined by $f'(x) = (x, f(x))$ is a polyhedral equivalence. **30**

1.6.11 Dimension of a polyhedron

The dimension of a polyhedron P is a defined to be Max. $\dim C$, $C \in \mathscr{P}$, where $\mathscr{P}$ is any polyhedral presentation of P.

OF course we have to check that this is independent of the presentation chosen. This follows from 1.5.2.

Let P and Q be two polyhedra and $f : P \to Q$ be a polyhedral map. Let $\gamma : P\times Q \to P$ and $\mu : P\times Q \to Q$ be the first and second projections. If $\mathscr{C}$ is any presentation of $\Gamma(f)$, then the open cells of the form $\lambda(C)$, $C \in \mathscr{C}$ is a presentation of P, regular if and only if $\mathscr{C}$ is regular. If f is a polyhedral equivalence, then the cells of the form $\mu(C)$, $C \in \mathscr{C}$ is a presentation of Q. This shows that

Proposition 1.6.12. *The dimension of a polyhedron is a polyhedral invariant.*

1.7 Refinement by bisection

Definition 1.7.1. If $\mathscr{P}$ and $\mathcal{Q}$ are polyhedral presentations, we say that $\mathscr{P}$ *refines* $\mathcal{Q}$, or $\mathscr{P}$ is a *refinement* of $\mathcal{Q}$ provided

(a) $|\mathscr{P}| = |\mathcal{Q}|$

(b) If $A \in \mathscr{P}$, and $B \in \mathcal{Q}$, then $A \cap B = \emptyset$ or $A \subset B$.

In otherwords, $\mathscr{P}$ and $\mathcal{Q}$ are presentations of the same polyhedron and each element (an open convex cell) of $\mathscr{P}$ is contained in each element of
31 $\mathcal{Q}$ which it intersects. Hereafter, when there is no confusion, we will refer to open convex cells and closed convex cells as *open cells* and *closed cells*. A polyhedral presentation is regular if and only if it refines itself.

Let $\mathscr{B} = (P; H^+, H^-)$ be a bisection of the ambient vector space V (1.3.9); $\mathfrak{a}$ a polyhedral presentation of a polyhedraon in V, and let $A \in \mathfrak{a}$. We say that $\mathfrak{a}$ *admits a bisection by* $\mathscr{B}$ *at* A*, provided:*

Whenever an open cell $A_1 \in \mathfrak{a}$ intersects ∂A (i.e. $A_1 \cap \partial A \neq \emptyset$), and $\dim A_1 < \dim A$, then either $A_1 \subset P$ or $A_1 \subset H^+$ or $A_1 \subset H^-$ (in particular this should be true for any cell in the boundary of A).

If $\mathfrak{a}$ admits a bisection by $\mathscr{B}$ at A, then we define a presentation $\mathfrak{a}'$ as follows, and call it the *result of bisecting* $\mathfrak{a}$ *by* $\mathscr{B}$ *at* A:

$\mathfrak{a}'$ consists of $\mathfrak{a}$ with the element A removed, and with the nonempty sets of the form, $A \cap P$ or $A \cap H^+$ or $A \cap H^-$, that is

$$\mathfrak{a}' = \Big\{(\mathfrak{a} - \{A\}) \cup \{A \cap P, A \cap H^+,\ A \cap H^-\} - \{\emptyset\}\Big\}.$$

By 1.4.13, and the definition of admitting a bisection, $\mathfrak{a}'$ is a polyhedral presentation. Clearly $\mathfrak{a}'$ refines $\mathfrak{a}$, if $\mathfrak{a}$ is regular.

We remark that it may well be the case that A is contained in P or H^+ or H^-. In this event, bisecting at A changes nothing at all, that is $\mathfrak{a}' = \mathfrak{a}$. If this is the case we call the bisection trivial. It is also possible, in the case of irregular presentations, that some or all of the sets $A \cap P$, $A \cap H^+$, **32**
$A \cap H^-$ may already be contained in $\mathfrak{a} - \{A\}$ in this event, bisection will not change as much as we might expect.

Ex. 1.7.2. Let A and B be two open cells, with $\dim A \le \dim B$ and $A \neq B$. Let $\mathscr{B}_j : \{P_j; H_j^+, H_j^-\}_{1 \le j \le m}$ be bisections of space such that A is the intersection of precisely one element from some of the $\mathscr{B}_j$'s. If $A \cap B \neq \emptyset$, then $\exists$ an ℓ, $1 \le \ell \le m$, such that $B \cap P$, $B \cap H^+$ and $B \cap H^-$ are all nonvacuous.

What we are aiming at is to show that every polyhedral presentation $\mathscr{P}$ has a regular refinement, which moreover is obtained from $\mathscr{P}$ by a particular process (bisections). The proof is by an obvious double induction; we sketch the proof below leaving some of the details to the reader.

Proposition 1.7.3. *REFI* $(\mathscr{P}, \mathscr{P}', \{S_i\})$

There is a procedure, which, applied to a polyhedral presentation $\mathscr{P}$, gives a finite sequences $\{S_i\}$ of bisections (at cells by bisections of space), which start on $\mathscr{P}$, give end result $\mathscr{P}'$, and $\mathscr{P}'$ is a regular presentation which refines $\mathscr{P}$.

Proof.

Step 1. First, we find a finite set $\mathscr{B}_j : \{P_j; H_j^+, H_j^-\}$, $j = 1, \dots n$ of bisections of the ambient space, such that every element of $\mathscr{P}$ is an intersection one element each from some of the $\mathscr{B}_j$'s. This is possible because every element of $\mathscr{P}$ is a finite intersection of hyperplanes and half spaces, and there are only a finite number of elements in $\mathscr{P}$.

33 **Step 2.** Write $\mathscr{P} = \mathscr{P}_0$. Index the cells of $\mathscr{P}_0$ in such a way that the dimension is a non decreasing function. That is define p_0 to be the cardinality of $\mathscr{P}_0$, arrange the elements of $\mathscr{P}_0$ as $D^0_0, \ldots, D^0_{p_0}$, such that $\dim D^0_k \leq \dim D^0_{k+1}$ for all $0 \leq k \leq p_0 - 1$.

Step 3. $S_{0,1}$ denotes the process of bisecting $\mathscr{P}_0$ at D^0_0 by $\mathscr{B}_1$. Inductively, we define $S_{0,k+1}$ to be the process of bisecting $\mathscr{P}_{0,k}$ at D^0_{k+1} by $\mathscr{B}_1$; and $\mathscr{P}_{0,k+1}$ the result. This is well defined since the elements of $\mathscr{P}_0$ are arranged in the order of nondecreasing dimension. This can be done until we get S_{0,p_0} and $\mathscr{P}_{0,p_0}$.

Step 4. Write $\mathscr{P}_1 = \mathscr{P}_{0,p_0}$, repeat step (2) and then the step (3) with bisection $\mathscr{B}_2$ instead of $\mathscr{B}_1$.

And so on until we get $\mathscr{P}_n$, when the process stops. $\mathscr{P}_n$ is clearly a refinement of $\mathscr{P} = \mathscr{P}_0$; it remains to show that $\mathscr{P}_n$ is regular. Each element by $\mathscr{P}_n$ belongs to some $\mathscr{P}_{i,j}$ and each element of $\mathscr{P}_n$ is a finite intersection of exactly one element each from a subfamily of the $\mathscr{B}_j$'s. It is easily shown by double induction that if $A \in \mathscr{P}_n$, then for any j, $i \leq j \leq n$, either $A \subset P_j$ or $A \subset H^+_j$ or $A \subset H^-_j$. That is $\mathscr{P}_n$ admits a bisection at A by $\mathscr{B}_j$ for any j, but the bisection is trivial. Let C, $D \in \mathscr{P}_n$, $C \neq D$ $C \cap D \neq \emptyset$, $\dim C \leq \dim D$. Then since C is an intersection of one element each from a subfamily of the $\mathscr{B}_j$'s, by 1.7.3, there exists an ℓ such that $D \cap P_\ell$, $D \cap H^+_\ell$ and $D \cap H^-_\ell$ are all nonempty.
34 But this is a contradiction. Hence $\mathscr{P}_n$ is regular. Write $\mathscr{P}_n = \mathscr{P}'$, $S_{i,j} = S_{p_0} + \cdots + p_i + j$. This gives the "REFI $(\mathscr{P}, \mathscr{P}', \{S_i\})$". $\square$

We can now draw a number of corollaries:

Corollary 1.7.4. *Any polyhedron has a regular presentation.*

Corollary 1.7.5. *Any two polyhedral presentations $\mathscr{P}$, Q of the same polyhedron X have a common refinement $\mathscr{R}$, which is obtained from $\mathscr{P}$ and from Q by a finite sequence of bisections.*

To see this, note that $\mathscr{P} \cup \mathcal{Q}$ is a polyhedral presentation of X. The application REFI $(\mathscr{P} \cup \mathcal{Q}, \mathscr{R}, \{S_i\})$ provides $\mathscr{R}$. Let $\{T_i\}$ and $\{U_k\}$ denote the subsequences applying to $\mathscr{P}$ and $\mathcal{Q}$ respectively; observe that they both result in $\mathscr{R}$.

Corollary 1.7.6. *Given any finite number $\mathscr{P}_1, \ldots, \mathscr{P}_r$ of polyhedral presentations, there is a regular presentation $\mathcal{Q}$ of $|\mathscr{P}_1| \cup \ldots \cup |\mathscr{P}_r|$, and $\mathcal{Q}$ has subpresentations $\mathcal{Q}_1, \ldots, \mathcal{Q}_r$, with $|\mathscr{P}_i| = |\mathcal{Q}_i|$ for all i and $\mathscr{P}_i$ is obtained from $\mathcal{Q}_i$ by a finite sequence of bisections.*

This is an application of REFI $(\mathscr{P}_1 \cup \ldots \cup \mathscr{P}_r, \mathcal{Q}, \{S_i\})$ and an analysis of the situation.

Chapter 2

Triangulation

As we have seen, every polyhedral presentation $\mathscr{P}$ has a regular refine- **35**
ment. This implies that any two polyhedral presentations of X have a common regular refinement, that if $X \subset Y$ are polyhedra there are regular presentations of Y containing subpresentations covering X, etc.. In this chapter we will see that in fact every polyhedral presentation has a simplicial refinement, and that given a polyhedral map $f : P \to Q$, there exist simplicial presentations of P and Q with respect to which f is "simplicial".

2.1 Triangulation of polyhedra

A simplicial presentation $\mathscr{S}$ of a polyhedron X is also known as a *linear triangulation* of X. We shall construct simplicial presentations from regular ones by "barycentric subdivision".

Definition 2.1.1. Let $\mathscr{P}$ be a regular presentation. A *centering* of $\mathscr{P}$ is a function $\eta : \mathscr{P} \to |\mathscr{P}|$, such that $\eta(C) \in C$, for every $C \in \mathscr{P}$.

In other words, a centering is a way to choose a point each from each element (an open convex cell) of $\mathscr{P}$.

Proposition 2.1.2. *If $C_0, C_1 \ldots C_k$ are elements of $\mathscr{P}$, ordered with respect to boundary relationship, then $\{\eta(C_0), \ldots, \eta(C_k)\}$ is an independent set for any centering η of $\mathscr{P}$.*

Proof. Immediate from 1.4.9, by induction. □

Proposition 2.1.3. *Suppose that $\mathscr{P}$ is a regular presentation and $C \in$*
36 *$\mathscr{P} \cdot C$ is the disjoint union of all open simplexes of the form*

$$0(\eta(A_0), \eta(A_1), \ldots, \eta(A_k), \eta(C))$$

where $A_i \in \mathscr{P}$, $A_0 < A_1 < \ldots < A_k$ and $A_i \subset \partial C$.

Proof. By induction. Assume the proposition to be true for cells of dimension $<\dim C$. ∂C is the union of all simplexes of the form $0(\eta(A_0), \eta(A_1), \ldots, \eta(A_k))$ where $A_i \in \mathscr{P}$, $A_i < C$ and $A_0 < \ldots < A_k$ (since $<$ is transitive). Since C is a bounded open convex cell C is the union of $0(\eta(C), x)$, $x \in \partial C$ and $\eta(C)$ (see the remark following 1.4.18). Now 2.1.2 completes the rest. □

It follows from 2.1.2 and 2.1.3, that if $\mathscr{P}$ is any regular presentation, then the set of all open simplexes of the form $0(\eta(C_0), \ldots, \eta(C_k))$, for $C_i \in \mathscr{P}$, with $C_0 < \ldots < C_k$, is a simplical presentation of $|\mathscr{P}|$. This leads to the following definition and proposition.

Definition 2.1.4. If $\mathscr{P}$ is any regular presentation, η a centering of $\mathscr{P}$; the *derived subdivision of $\mathscr{P}$ relative to η* is the set of open simplexes of the form $0(\eta(C_0), \ldots, \eta(C_k))$, $C_i \in \mathscr{P}$, $C_0 < \ldots < C_k$. It is a simplicial presentation (of $|\mathscr{P}|$) and is denoted by $d(\mathscr{P}, \eta)$.

The vertices of $d(\mathscr{P}, \eta)$ are precisely the points (0-cells) $\eta(C)$, $C \in \mathscr{P}$. When η is understood, or if the particular choice of η is not so important, we refer to $d(\mathscr{P}, \eta)$ as a derived subdivision of $\mathscr{P}$ and denote it by $d\mathscr{P}$.

Proposition 2.1.5. *Every polyhedral presentation admits of a simplicial refinement.*

37 Hence every polyhedron can be triangulated.

2.2 Triangulation of maps

Now, we return to polyhedral maps. If $f : P \to Q$ is a polyhedral map, we have seen that the map $f' : P \to \Gamma(f)$ given by $f'(x) = (x, f(x))$ is a polyhedral equivalence and that any presentation of $\Gamma(f)$ gives a presentation of P by linear projection. Also, we saw in 1.3, that if A is a convex subset of vector space V and $\varphi : A \to W$ a map of A into a vector space W, φ is linear if and only if the graph of φ is convex. Combining these two remarks, we have that a polyhedral map is 'piecewise linear' or as Alexander called it 'linear in patches'.

Next, an attempt to describe polyhedral maps in terms of presentations of polyhedra leads to the following definition.

Definition 2.2.1. Let $\mathfrak{a}$ and $\mathscr{B}$ be regular presentations. A function $\varphi : \mathfrak{a} \to \mathscr{B}$ is called *combinatorial* if for all $A_1, A_2 \in \mathfrak{a}$, $A_1 \leq A_2$ implies $\varphi(A_1) \leq \varphi(A_2)$.

But unfortunately there may be several distinct polyhedral maps $|\mathfrak{a}| \to |\mathscr{B}|$ inducting the same combinatorial map $\mathfrak{a} \to \mathscr{B}$, and a map $|\mathfrak{a}| \to |\mathscr{B}|$ inducing some combinatorial map $\mathfrak{a} \to \mathscr{B}$ need not even be polyhedral (We will see more of these when we come to 'standard mistake'). If turns out that a map $\mathfrak{a} \to \mathscr{B}$ induces a unique map $|\mathfrak{a}| \to |\mathscr{B}|$ if we require that the induced map to be linear on each cell of $\mathfrak{a}$. But in this case it is sufficient to know the map on 0-cells (vertices); one can extend by linearly. This naturally leads to simplicial maps.

Definition 2.2.2. Let X and Y be polyhedra, $\mathscr{S}$ and $\mathscr{Z}$ simplicial pre- **38**
sentations of X and Y respectively. A map $f : X \to Y$ is said to be *simplicial with respect to $\mathscr{S}$ and $\mathscr{Z}$*, iff

(1) f maps vertices of each simplex in $\mathscr{S}$ into the vertices of some simplex in $\mathscr{Z}$.

and

(2) f is linear on the closure of each simplex in $\mathscr{S}$.

f is polyhedral, since its graph has a natural simplicial presentation isomorphic so $\mathscr{S}$.

Let $\mathscr{S}$ and $\mathscr{Z}$ be two simplicial presentations. Let $\mathscr{S}_0$ (resp. $\mathscr{Z}_0$) be the set of vertices of $\mathscr{S}$ (resp. $\mathscr{Z}$). If $\mathscr{L} : \mathscr{S}_0 \to \mathscr{Z}_0$ is a map, which carries the vertices of a simplex of $\mathscr{S}$ is a *simplicial* map from $\mathscr{S}$ to $\mathscr{Z}$.

Example 2.2.3. If $\varphi : \mathscr{P} \to \mathcal{Q}$ is a combinatorial map η, θ centerings of $\mathscr{P}$ and $\mathcal{Q}$ respectively, the map which carries ηC to $\theta(\varphi(C))$ is a simplicial map from $d(\mathscr{P}, \eta)$ to $d(\mathcal{Q}, \theta)$.

We now proceed to show that every polyhedral map is simplicial with respect to some triangulations.

Let P and Q be two polyhedra and $f : P \to Q$ be a polyhedral map. Let $\mathscr{P}$, $\mathcal{Q}$ and $\mathscr{C}$ be presentations of P, Q and $\Gamma(f) \subset P \times Q$. Let $\mathfrak{a}$ be a regular presentation of $P \times Q$ which refines $(\mathscr{P} \times \mathcal{Q}) \cup \mathscr{C}$, and let $\mathscr{C}'$ be the subpresentation of $\mathfrak{a}$ which covers $\Gamma(f)$.

39 Let λ and μ be the projections of $P \times Q$ onto P and Q respectively. By the refinement process there is a regular presentation $\mathcal{Q}'$ of Q refining $\mathcal{Q}$ such that:

(*) If $A \in \mathcal{Q}'$, $C \in \mathscr{C}'$, $A \cap \mu(C) \neq \emptyset$, then $A \subset \mu(C)$. Then, if $C \in \mathscr{C}'$, $\mu(C)$ is the union of elements of $\mathcal{Q}'$.

Now we look at the presentations $\mathscr{C}'' = \mathscr{C}' \cdot (\mathscr{P} \times \mathcal{Q}')$ of $\Gamma(f)$. The cells of $\mathscr{C}''$ are by definition of the form $C'' = C \cap (A \times B')$, $C \in \mathscr{C}'$, $A \in \mathscr{P}$, $B' \in \mathcal{Q}'$. Clearly $C'' \subset C \cap \mu^{-1}(B')$. On the other hand, since $\mathcal{Q}'$ is a refinement of $\mathcal{Q}$, there is an open cell $B \in \mathcal{Q}$ with $B \supset B'$. Since $\mathscr{C}'$ is a subpresentation of a refinement $\mathfrak{a}$ of $\mathscr{P} \times \mathcal{Q}$, if $C'' \neq 0$, $C \subset A \times B$. Hence if $(x, y) \in C \cap \mu^{-1}(B')$, then $x \in A$, $y \in B'$, so $(x, y) \in A \times B'$. Hence $C \cap \mu^{-1}(B') \subset C \cap (A \times B') = C''$. Thus $C'' = C \cap \mu^{-1}(B')$. Hence $\mathscr{C}''$ can be also described as

$$\mathscr{C}'' = \left\{C \cap \mu^{-1}(B') | C \cap \mu^{-1}(B') \neq 0,\ C \in \mathscr{C}',\ B' \in \mathcal{Q}'\right\}$$

Now, clearly $\mathscr{P}' = \lambda(\mathscr{C}'') = \{\lambda(D) | D \in \mathscr{C}''\}$ is a regular presentation of $P (\lambda | \Gamma(f)$ is $1-1$ and λ is linear) with reference to the ambient vector spaces. Now the claim is that f induces a combinatorial map $\mathscr{P}' \to \mathcal{Q}'$. Let A be any cell of $\mathscr{P}' \cdot (\lambda | \Gamma(f))^{-1}(A)$ is a cell of $\mathscr{C}''$, say some $C \cap \mu^{-1}(B')$. $f(A) = \mu(C \cap \mu^{-1}(B')) = \mu(C) \cap B' = B'$ by (*). Thus $f(A) \in \mathcal{Q}'$. $\partial(C \cap \mu^{-1}(B'))$ is the union of $\partial C \cap \mu^{-1}(B')$, $C \cap \mu^{-1}(\partial B')$ and $\partial C \cap \mu^{-1}(\partial B')$; (by 1.4.5) and so $\mu(\partial(C \cap \mu^{-1}(B')))$ is the union of

$\mu(\partial C) \cap B'$, $\mu(C) \cap \partial B'$, and $\mu(\partial C) \cap B'$, hence $\mu(\partial C) \subset \overline{B}^1$. Hence if $A_1 \leq A$, $f(A_1) \leq B'$. Thus f induces a combinatorial map from $\mathscr{P}'$ to 40
$\mathcal{Q}'$. Moreover, since the presentation $\mathscr{P}'$ comes from $\mathscr{C}''$, the graph of f restricted to the closure of each cell of $\mathscr{P}'$ is a closed cell, and hence f is linear on the closure of each cell of $\mathscr{P}'$.

The discussion so far can be summarized as:

Theorem 2.2.4. *Let $f : P \to Q$ be a polyhedral map, and let $\mathscr{P}$, $\mathcal{Q}$ be polyhedral presentations of P and Q respectively. Then there exist regular refinements $\mathscr{P}'$ and $\mathcal{Q}'$ of $\mathscr{P}$ and $\mathcal{Q}$ such that*

(1) If $A \in \mathscr{P}'$, $f(A) \in \mathcal{Q}'$. The induced map from $\mathscr{P}'$ to $\mathcal{Q}'$ is combinatorial.

(2) f is linear on the closure of each cell of $\mathscr{P}'$.

Furthermore,

2.2.5 If $\mathscr{P}$ and $\mathcal{Q}$ are regular and if there is a regular presentation $\mathscr{C}$ of $\Gamma(f)$ such that

(a) For each $C \in \mathscr{C}$, $\lambda(C)$ is contained in some element of $\mathscr{P}$,

(b) For each $C \in \mathscr{C}$, $\mu(C)$ is the union of elements of $\mathcal{Q}$,

then in the above theorem we can take $\mathcal{Q}' = \mathcal{Q}$ (in other words, a combinatorial map can be found refining only $\mathscr{P}$, not $\mathcal{Q}$).

To apply 2.2.4 to the problem of simplicial maps, we can use 2.2.3 as follows: First we choose some centering θ of $\mathcal{Q}'$, and then a centering η of $\mathscr{P}'$ so that

$$f(\eta(C)) = \theta(f(C)) \quad \text{for all} \quad C \in \mathscr{P}'.$$

Since f is linear an each element of $\mathscr{P}'$, we have that $f : P \to Q$ is 41
simplicial with respect to $d(\mathscr{P}', \eta)$ and $d(\mathcal{Q}', \theta)$. Hence,

Corollary 2.2.6. *Given a polyhedral map $f : P \to Q$, there exist triangulations $\mathscr{S}$ and $\mathscr{Z}$ of P and Q, with respect to which f is simplicial. Moreover, $\mathscr{S}$ and $\mathscr{Z}$ can be chosen to refine any given presentations of P and Q.*

Defining the *source* and *target* of a map $f : K \to L$ to be K and L respectively. We may now state a more general result, details of the proof left as an exercise.

Theorem 2.2.7. *Let $\{K_\alpha\}$ be a finite set of polyhedra, with $\mathscr{L} = 1, \dots, n$; let $f_r : K_{\mathscr{L}_r} \to K_{\beta_r}$ be a finite set of polyhedral maps, the sources and targets being all in the given set of polyhedra. Suppose that for each γ, $\mathscr{L}_\gamma < \beta_\gamma$, and each $K_{\mathscr{L}}$ occurs as the source of at most one of the maps f (i.e. $\gamma \neq \delta$ implies $\mathscr{L}_\gamma \neq \mathscr{L}_\delta$). Let $\mathscr{P}_\gamma$ be a presentation of K_γ for each γ. Then there is a set of simplicial presentations $\{\mathscr{S}_\gamma\}$, with $\mathscr{S}_\gamma$ refining $\mathscr{P}_\gamma$, such that for all γ, f_γ is simplicial with reference to*

$$\mathscr{S}_{\mathscr{L}_\gamma} \quad \textit{and} \quad \mathscr{S}_{\beta_\gamma}$$

That is to say, the whole diagram $\{f_\gamma\}$ can be triangulated.

The condition on sources is not always necessary, for example:

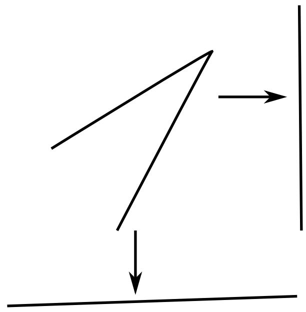
Ex. 2.2.8. A diagram of polyhedral maps

$$\begin{tikzcd} & Y \\ X \arrow[ur, "f"] \arrow[dr] & \\ & Z \end{tikzcd}$$

42 can be triangulated if $f : X \to Y$ is an imbedding.

However

Ex. 2.2.9. The following diagram of polyhedral maps (each map is a linear projection)



cannot be triangulated.

Ex. 2.2.10. Let $\mathscr{P}$ be a presentation of a polyhedron P in V. $\varphi : V \to W$ be a linear map, then $\varphi(\mathscr{P}) = \{\varphi(C) | C \in \mathscr{P}\}$ is a presentation of $\varphi(P)$.

Ex. 2.2.11. Let $f : P \to Q$ be a polyhedral map, and X a subpolyhedron of P. Then $\dim f(X) \leq \dim X$.

Ex. 2.2.12. If $f : P \to Q$ is a polyhedral map Y is a subpolyhedron of Q, $f^{-1}(Y)$ is a subpolyhedron of X.

Next, one can discuss abstract simplicial complexes, their geometric realizations etc. We do not need them until the last chapter. The reader is referred to Pontryagin's little book mentioned in the first chapter for these things.

Chapter 3

Topology and Approximation

Since we know that intersection and union of two polyhedra is a polyhe- 43
dron, we may define a topology on a polyhedron X, by describing sets of the form $X - Y$, for Y a subpolyhedron, as a basis of open sets. If, one the other hand, X is a polyhedron in a finite dimensional real vector space V, then V has various Euclidean metrics (all topologically equivalent) and X inherits a metric topology.

Ex. These topologies on X are equal.

The reason is that any point of V is contained in an arbitrary small open cell, of the same dimension as V.

It is easy to see tht a closed simplex with this topology is compact. Hence every polyhedron, being a finite union of simplexes is compact. The graph of a polyhedral map is then compact, and hence f is continuous. Thus we have an embedding of the category of polyhedra and polyhedral maps into the category of compact metric spaces and continuous maps.

It is with respect to any metric giving this topology that our approximation theorems are phrased.

A polyhedron is an absolute neighbourhood retract, and the results that we have are simply obtained from a hard look at such results for A.N.R's.

It turns our that we obtain a version of the simplicial approximation 44
theorem, which was the starting point, one may say, of the algebraic

topology of the higher dimensional objects. The theorem has been given a ‘relative form’ by Zeeman, and we shall explain a method which will give this as well as other related results.

We must first say something about polyhedral neighbourhoods.

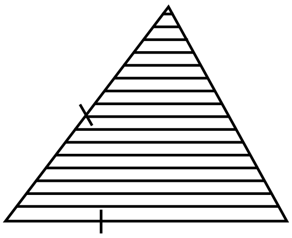
3.1 Neighbourhoods that retract

Let $\mathscr{P} \subset \mathcal{Q}$ be regular presentations. Consider the open cells C of $\mathcal{Q}$, with $\overline{C} \cap |\mathscr{P}| \neq \emptyset$, together with A, $A < C$, $A \in \mathcal{Q}$, for such C. The set of all these open cells is a subpresentations $\mathscr{N}$ of $\mathcal{Q}$. $|\mathscr{N}|$ is a neighbourhood of $|\mathscr{P}|$ in $|\mathcal{Q}|$. For, if $\mathscr{N}'$ is the set of cells $C' \in \mathcal{Q}$ such that $\overline{C}' \cap |\mathscr{P}| = \emptyset$, then $\mathscr{N}'$ is a subpesentation of $\mathcal{Q}$ and $|\mathcal{Q}| - |\mathscr{N}'| \subset |\mathscr{N}|$. If $\mathcal{Q}$ is simplicial, $\mathscr{N}$ can be described as the subpresentation, consisting of open simplexes of $\mathcal{Q}$ with some vertices in $\mathscr{P}$ together with their faces.

If $\mathscr{P} \subset \mathcal{Q}$ is a subpresentation, we say that $\mathscr{P}$ is full in $\mathcal{Q}$; if for every $C \in \mathcal{Q}$ either $\overline{C} \cap |\mathscr{P}| = \emptyset$ or there is a $A \in \mathscr{P}$ with $\overline{C} \cap |\mathscr{P}| = \overline{A}$.

In the case of simplicial presentations, this is the same as saying that if an open simplex σ of $\mathcal{Q}$ has all its vertices in $\mathscr{P}$, then σ itself is in $\mathscr{P}$.

An example of a nonfull subpresentation:



45 **3.1.1** If $\mathscr{P} \subset \mathcal{Q}$ are regular presentations, then $d\mathscr{P}$ is full in $d\mathcal{Q}$.

For, if η is any centering, then an element (an open simplex) of $d\mathcal{Q}$ is of the form $0(\eta(C_0)), \ldots, \eta(C_k))$, $C_k \in \mathcal{Q}$, $C_0 < \ldots < C_k$. If C_ℓ, $0 \leq \ell \leq k$, is the last element of the C_i's that is in $\mathscr{P}$, then C_j, $j \leq \ell$, are necessarily in $\mathscr{P}$. Then $0(\eta(C_0), \ldots, \eta(C_\ell)) \in d\mathscr{P}$, and $\overline{0}(\eta(C_0), \ldots, \eta(C_k)) \cap |d\mathscr{P}| = \overline{0}(\eta(C_0), \ldots, \eta(C_\ell))$.

Definition 3.1.2. If $\mathscr{P}$ is full in Q, the *simplicial neighbourhood of $\mathscr{P}$ in Q*, is the subpresentation of dQ consisting of all simplexes of dQ whose vertices $\eta(C)$ are centers of cells C of Q with $\overline{C} \cap |\mathscr{P}| \neq \emptyset$. It is denoted by $N_Q(\mathscr{P})$ (or $N_Q(\mathscr{P}, \eta)$ when we want to make explicit the centering).

Clearly $N_Q(\mathscr{P})$ is a full subpresentation of dQ. It can be also described as the set of elements σ of dQ, for which $\overline{\sigma} \cap |d\mathscr{P}| = \overline{\sigma} \cap |\mathscr{P}| = \emptyset$ plus the faces of such σ. Hence $|N_Q(\mathscr{P})|$ is a neighbourhood of $|\mathscr{P}|$ in the topological sense.

Such a neighbourhood as $|N_Q(\mathscr{P})|$ of $|\mathscr{P}|$ is usually referred to as a '*second derived neighbourhood*' of $|\mathscr{P}|$ in $|Q|$, for the following reason: If $X \subset Y$ are polyhedra; to get such a neighbourhood we first start with a regular presentation $\mathfrak{a}$ of Y containing a subpresentation $\mathscr{B}$ covering X, derive once so that $d\mathscr{B}$ is full in $d\mathfrak{a}$, then derive again and take $|N_{d\mathfrak{a}}(d\mathscr{B})|$.

Now we can define a simplicial map $r : N_Q(\mathscr{P}) \to d\mathscr{P}$, using the **46**
property of fullness of $\mathscr{P}$ in Q. If $C \in Q$, with $\overline{C} \cap |\mathscr{P}| \neq \emptyset$, we know that there is a $A \in \mathscr{P}$, such that $\overline{C} \cap |\mathscr{P}| = \overline{A}$, and this A is uniquely determined by C. We define $r(\eta C) = \eta A$.

Ex. 3.1.3. The map r thus defined is a simplicial retraction of $N_Q(\mathscr{P})$ onto $d\mathscr{P}$.

That is r is a simplicial map from $N_Q(\mathscr{P})$ to $d\mathscr{P}$, which when restricted to $d\mathscr{P}$ is identity. r defines therefore a polyhedral map, which also we shall call $r : |N_Q(\mathscr{P})| \to |d\mathscr{P}|$. We have proved

3.1.4 If X is a subpolyhedron of Y, there is a polyhedron N which is a neighbourhood of X in Y, and there is a polyhedral retraction $r : N \to X$.

3.2 Approximation Theorem

We imagine our polyhedra to be embedded in real vector spaces (we have been dealing only with euclidean polyhedra) with euclidean metrics. Let X, Y be two polyhedra, ρ, ρ' be metrics on X and Y respectively coming from the vector spaces in which they are situated. If α,

$\beta : Y \to X$ are two functions, we define

$$\rho(\alpha, \beta) = \operatorname*{Sup}_{x \in Y} \rho(\alpha(x), \beta(x))$$

If A is a subset of X, we define $\operatorname{diam} A = \sup_{x,y \in A} \rho(x, y)$, and if B is a subset of Y, we define $\operatorname{diam} B = \operatorname*{Sup}_{x,y \in B} \rho'(x, y)$.

47 We can consider X to be contained in a convex polyhedron Q. If X is situated in the vector space V, we can take Q to be large cube or the convex hull of X. Let N be a second derived neighbourhood of X in Q and $r : N \to X$ be the retraction. Now Q being convex and N a neighbourhood of X and Q, for any sufficiently small subset S of X, $K(S) \subset N$ (recall that $K(S)$ denotes the convex hull of S). This can be made precise in terms of the metric; and is a uniform property since X is compact. Next observe that we can obtain polyhedral presentations $\mathscr{P}$ of X, such that diameter of each element of $\mathscr{P}$ is less than a prescribed positive number. This follows for example from refinement process. Now theorem is

Theorem 3.2.1. *Given a polyhedron X, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any pair of polyhedra $Z \subset Y$, and any pair of functions $f : Y \to X$, $g : Z \to X$ with f continuous and g polyhedral, if $\rho(f|Z, g) < \delta$, then there exists a $\overline{g} : Y \to X$, $\overline{g}$ polyhedral, $\overline{g}|Z = g$, and $\rho(f, \overline{g}) < \epsilon$.*

Proof. We embed X in a convex polyhedron Q, in which there is a polyhedral neighbourhood N and a polyhedral retraction $r : N \to X$ as above. It is clear from the earlier discussion, that given $\epsilon > 0$, there is a $\eta > 0$, such that if a set $A \subset X$ has diameter $< \eta$, then $K(A) \subset N$ and diamter $r(K(A)) < \epsilon$. Define $\delta = \eta/3$.

Now because of the uniform continuity of f, (Y is compact), there
48 is a $\theta > 0$, such that if $B \subset Y$ and diam. $(B) < \theta$, then diam $f(B) < \delta$.

From this it follows that, still assuming $B \subset Y$, and diameter $B < \theta$, and additionally that $\rho(f|Z, g) < \delta$; that the set $f(B) \cup g(B \cap Z)$ has diameter less than $3\delta = \eta$. And hence we know that

$$(*) \qquad \begin{cases} K(f(B) \cup g(B \cap Z)) \subset N, & \text{and} \\ \operatorname{diam} r(K(f(B) \cup g(B \cap Z))) < \epsilon. \end{cases}$$

Then we find a presentation $\mathscr{S}$ of Y, such that the closure of every element of $\mathscr{S}$ has diameter less than θ. Also there is a presentation $\mathscr{Z}$ of Z on the closure of every element of which g is linear. Refining $\mathscr{S} \cup \mathscr{Z}$ and taking derived subdivisions (still calling the presentations covering Y and Z, as $\mathscr{S}$ and $\mathscr{Z}$ respectively), we have the following situation:

$\mathscr{Z} \subset \mathscr{S}$, are simplicial presentations of $Z \subset Y$, on each closed $\mathscr{Z}$-simplex g is linear, the diameter of each closed $\mathscr{S}$-simplex $< \theta$.

We now define $h : Y \to Q$ as follows: On a 0-simplex v of $\mathscr{Z}$, $h(v) = g(v)$. On a 0-simplex w of $\mathscr{S} - \mathscr{Z}$, $h(w) = f(w)$. Extend h linearly on each simplex, this is possible since Q is convex. But now, if $\overline{\sigma} = [v_0, \ldots, v_n]$ is the closure of a $\mathscr{S}$-simplex, then $h(\overline{\sigma}) \subset K(f(\overline{\sigma}) \cup g(\overline{\sigma} \cap Z)) \subset N$; this is a computation made above (*) since diam. $\overline{\sigma} < \theta$.

And so $h(Y) \subset N$. Also it is the case that h is polyhedral, since h is liner on the closure of each simplex of $\mathscr{S}$, and on $|\mathscr{Z}| = Z$, clearly, h agrees with g.

Define, $\overline{g} : Y \to X$ to be $r \circ h$. Since r and h are polyhedral so is $\overline{g}$; 49
since $h|Z = g$ and r is identity on X; it follows that $\overline{g}|Z = g$. To compute $\rho(\overline{g}, f)$ we observe that any $y \in Y$ is contained in some closed simplex $\overline{\sigma}$, $\sigma \in \mathscr{S}$, and both $f(y)$ and $h(y)$ are contained in $K(f(\overline{\sigma}) \cup g(\overline{\sigma} \cap Z))$; and hence both $f(y)$ and $g(y)$ are contained in

$$r(K(f(\overline{\sigma}) \cup g(\overline{\sigma} \cap Z)))$$

This set by (*) has diameter $< \epsilon$. Hence $\rho(\overline{g}, f) < \epsilon$. □

We now remark a number of corollaries:

Corollary 3.2.2. *Let X, Y, Z be polyhedra, $Z \subset Y$, and $f : Y \to X$ a continuous map such that $f|Z$ is a polyhedral. Then f can be approximated arbitrarily closely by polyhedral maps $g : Y \to X$ such that $g|Z = f|Z$.*

The next is not a corollary of 3.2.1, (it could be) but follows from the discussion there.

3.2.3 Any two continuous maps $f_1, f_2 : Y \to X$, if they are sufficiently close are homotopic. (Also how close depends only on X, not Y or the maps involved). If f_1 and f_2 are polyhedral, we can assume the homotopy also to be polyhedral, and fixed on any sub-polyhedron on which f_1 and f_2 agree.

Proof. Let N and X be as before. Let η be a number such that if $A \subset X$, $\operatorname{diam} A < \eta$, then $K(A) \subset N$. If $\rho(f_1, f_2) < \eta$, then $F(y, t) = t f_1(y) + (1 - t) f_2(y) \in N$, for $0 \leq t \leq 1$ and all $y \in Y$ and $r \cdot F$, where $r : N \to X$ is the retraction, gives the required homotopy. If f_1, f_2 are polyhedral, we can apply 3.2.1 to obtain a polyhedral homotopy with the desired properties. □

50 **Remark.** The above homotopies are small in the sense, that the image of x is not moved too far from $f_1(x)$ and $f_2(x)$.

3.2.4 Homotopy groups and singular homology groups of a polyhedron can be defined in terms of continuous functions or polyhedral maps from closed simplexes into X. The two definitions are naturally isomorphic. The same is true for relative homotopy groups, triad homotopy groups etc.

The corollary 3.2.2 is Zemman's version of the relative simplicial approximation theorem. From this (coupled with 4.2.13) one can deduce (see M. Hirsch, "A proof of the nonretractibility of a cell onto its boundary", Proc. of A.M.S., 1936, Vol. 14), Brouwer's theorems on the noncontractibility of the n-sphere, fixed point property of the n-cell, etc. It should be remarked that the first major use of the idea of simplicial approximation was done by L.E.J. Brouwer himself; using this he defined degree of a map, proved its homotopy invariance, and incidentally derived the fixed point theorem.

It should be remarked that relative versions of 3.2.1 are possible. For example define a pair (X_1, X_2) to be a space (or a polyhedron) and a subspace (or a subpolyhedron) and continuous (or polyhedral) maps $f : (X_1, X_2) \to (Y_1, Y_2)$ to be the appropriate sort of function $X_1 \to X_2$ which maps X_2 into Y_2. Then Theorem 3.2.1 can be stated in terms of pairs and the proof of this exactly the same utilising modifications of 3.1.4 and the remarks at the beginning of 3.2 which are valid for pairs.

51 Another relative version of interset is the notion of *polyhedron over*

A, that is, a polyhedral map $\mathcal{L} : X \to A$. A map $f : (\alpha : X \to A) \to (\beta : Y \to A)$ is a function $f : X \to Y$ such that $\alpha = \beta \cdot f$; we can consider either polyhedral or continuous maps. The reader should state and prove 3.2.1 in this context (if possible).

3.3 Mazur's criterion

We shall mention another result (see B. Mazur "The definition of equivalence of combinatorial imbeddings" Publications Mathematiques, No.3, I.H.E.S., 1959) at this point, which shows that, in a certain sense, close approximations to embeddings are embeddings (in an ambient vector space).

Let $\mathscr{X}$ be a simplicial presentation of X, and let V be a real vector space. Let $\mathscr{X}_0$ denote the set of vertices of $\mathscr{X}$. Given a function $\varphi : \mathscr{X}_0 \to V$, we can define an extension $\widetilde{\varphi} : |\mathscr{X}| \to V$ by mapping each simplex linearly. Clearly if $Y \subset V$ is any polyhedron containing $\widetilde{\varphi}(X)$, the resulting map $X \to Y$ is polyhedral. We call $\widetilde{\varphi}$ the linear extension of φ. $\widetilde{\varphi}$ is called an *embedding* if it maps distinct points of X into distinct points in V.

3.3.1 (Mazur's criterion for non-embeddings)

If the linear extension $\widetilde{\varphi}$ of $\varphi : \mathscr{X}_0 \to V$ is not an embedding, then there are two open simplexes σ and τ of $\mathscr{X}$, with no vertices in common, such that $\widetilde{\varphi}(\sigma) \cap \widetilde{\varphi}(\tau) \neq \emptyset$.

Proof. The proof is in two stages.

(A) If $\sigma = 0(v_0, \ldots, v_n)$ and $\{\varphi(v_0), \ldots, \varphi(v_n)\}$ is not independent, then there are faces σ_1 and σ_2 of σ, without vertices in common, such **52**
that $\widetilde{\varphi}(\sigma_1) \cap \widetilde{\varphi}(\sigma_2) \neq \emptyset$ (This is just 1.2.6).

(B) Thus we can assume that for every σ of $\mathscr{X}$, $\widetilde{\varphi}(\sigma)$ is also an open simplex of the same dimension. Consider pairs of distinct open simplexes $\{\rho, \rho'\}$ such tht $\widetilde{\varphi}(\rho) \cap \widetilde{\varphi}(\sigma') \neq \emptyset$. Let $\{\sigma, \tau\}$ be such a pair, which in addition has the property $\dim \sigma + \dim \tau$ is minimal among such pairs. We can now show that σ and τ have no vertex in common. If $\sigma = 0(v_0, \dots, v_m)$ and $\tau = 0(w_1, \dots, w_n)$, then if $\widetilde{\varphi}(\sigma) \cap \widetilde{\varphi}(\tau) \neq 0$, there is an equation

$$r_0\varphi(v_0) + \cdots + r_m\varphi(v_m) = s_0\varphi(w_1) + \cdots + s_n\varphi(w_n)$$

with $r_0 + \cdots + r_m = 1 = s_0 + s_1 \cdots + s_n$. Here r_i and s_i are strictly greater than 0, for otherwise $\dim\sigma + \dim\tau$ will not be minimal.

Now if and have a common vertex, say, for example, $v_0 = w_0$, and $r_0 \geq s_0$, we can write

$$(r_0 - s_0)\varphi(v_0) + \sum_{i\geq 1} r_i\varphi(v_i) = \sum_{j\geq 1} s_j\varphi(w_j).$$

Multiplying by $(1 - s_0)^{-1}$, we see that some face of $\widetilde{\varphi}(\sigma)$ intersects a proper face $\widetilde{\varphi}(0(w_1), \dots, w_n))$ of $\widetilde{\varphi}(\sigma)$. So that σ and τ had not the minimal dimension compatible with the properties $\sigma \neq \tau, \widetilde{\varphi}(\sigma) \cap \widetilde{\varphi}(\tau) \neq \emptyset$. □

Now it easily follows, since to check $\widetilde{\varphi}$ is an embedding we need only check that finitely many compact pairs $\{(\widetilde{\varphi}(\overline{\sigma}), \widetilde{\varphi}(\overline{\tau})), \overline{\sigma} \cap \overline{\tau} = \emptyset\}$ do not intersect;

53 **Proposition 3.3.2.** *Let $\mathscr{X}$ be a simplicial presentation of X contained in a vector space V, let $\mathscr{X}_0$ be the set of vertices. Then there exists an $\epsilon > 0$, such that if $\varphi : \mathscr{X}_0 \to V$ is any function satisfying $\rho(v, \varphi(v)) < \epsilon$ for all $v \in \mathscr{X}_0$, then the linear extension $\widetilde{\varphi} : X \to V$ is an embedding.*

This is a sort of stability theorem for embeddings, that is, if we perturb a little the vertices of an embedded polyhedron, we still have an embedding.

Chapter 4

Link and Star Technique

4.1 Abstract Theory I

Definition 4.1.1 (Join of open simplexes)**.** Suppose σ and two open sim- 54
plexes in the same vector space. We say that $\sigma\tau$ *is defined*, when

(a) the sets of vertices of σ and τ are disjoint

(b) the union of the set of vertices of σ and τ is independent.

In such a case we define $\sigma\tau$ to be the open simplex whose set of vertices is the union of those of σ and of τ. If σ is a 0-simplex, we will denote $\sigma\tau$ by $\{x\}\tau$ or $\tau\{x\}$ where x is the unique point in σ.

We also, by convention, where σ (or τ) is taken to be the empty set $\emptyset$, make the definition

$$\emptyset\sigma = \sigma\emptyset = \sigma$$

Clearly $\dim\sigma\tau = \dim\sigma + \dim\tau + 1$, even when one or both of them are empty.

Ex. 4.1.2. $\sigma\tau$ is defined if and only if $\overline{\sigma} \cap \overline{\tau} = \emptyset$, and any two open intervals $O(x, y)$, $O(x', y')$ are disjoint, where x, $x' \in \overline{\sigma}$, y, $y' \in \overline{\tau}$, $x \neq x'$ or $y \neq y'$. In this case $\sigma\tau$ is the union of open 1-simplexes $O(x, y)$, $x \in \sigma, y \in \tau$.

This is easy. Actually it is enough to assume $O(x, y) \cap O(x', y') = \emptyset$
for x, $x' \in \sigma$, y, $y' \in \tau$; $x \neq x'$ or $y \neq y'$. That it is true for points of $\overline{\sigma}$ 55

and $\overline{\tau}$ and $\overline{\sigma} \cap \overline{\tau} = \emptyset$ follow from this.

Ex. 4.1.3. When $\sigma\tau$ is defined, the faces of $\sigma\tau$ are the same as $\sigma'\tau'$, where σ' and τ' are faces of σ and τ respectively. If either $\sigma' \neq \sigma$ or $\tau' \neq \tau$, then $\sigma'\tau'$ is a proper face of $\sigma\tau$.

Ex. 4.1.4. Let σ and τ be in ambient vector spaces V and W. In $V \times W \times \mathbb{R}$, let $\widetilde{\sigma} = \sigma \times 0 \times 0$ and $\overline{\tau} = 0 \times \tau \times 1$. Then $\widetilde{\sigma}\widetilde{\tau}$ is defined.

Definition 4.1.5. Let $\mathscr{S}$ be a simplicial presentation, and σ an element of $\mathscr{S}$. Then the *link of σ in $\mathscr{S}$* denoted by $Lk(\sigma, \mathscr{S})$ is defined as

$$Lk(\sigma, \mathscr{S}) = \{\tau \in \mathscr{S} | \sigma\tau \quad \text{is defined.}\}$$

$$Lk(\sigma, \mathscr{A}) = \mathscr{S} \quad \text{if} \quad \sigma = \emptyset.$$

Obviously $Lk(\sigma, \mathscr{S})$ is a subpresentation of $\mathscr{S}$.

In case σ is 0-dimensional, we write $Lk(x, \mathscr{S})$ for $Lk(\sigma, \mathscr{S})$ where x is the unique element in σ.

Ex. 4.1.6. If $\tau \in Lk(\sigma, \mathscr{S})$, then

$$Lk(\tau, Lk(\sigma, \mathscr{S})) = Lk(\sigma\tau, \mathscr{S}).$$

Notation 4.1.7. If σ is an open simplex, then $\{\overline{\sigma}\}$ and $\{\partial\sigma\}$ will denote the simplicial presentations of $\overline{\sigma}$ and $\partial\sigma$ made up of faces of σ.

Ex. 4.1.8. If $\tau = \rho\sigma$, and $\dim \rho \geq 0$, then

$$Lk(\rho, \{\partial\tau\}) = \{\partial\sigma\}$$
$$Lk(\rho, \{\overline{\tau}\}) = \{\overline{\sigma}\}.$$

56 **Definition 4.1.9.** Let $\mathfrak{a}$ and $\mathscr{B}$ be simplicial presentations such that for all $\sigma \in \mathfrak{a}$, $\tau \in \mathscr{B}$, $\sigma\tau$ is defined, and $\sigma\tau \cap \sigma'\tau' = \emptyset$ if $\sigma \neq \sigma'$ or $\tau \neq \tau'$. Then we say that the *join of* $\mathfrak{a}$ and $\mathscr{B}$ *is defined*, and define the *join of* $\mathfrak{a}$ *and* $\mathscr{B}$, denoted by $\mathfrak{a} * \mathscr{B}$ to be the set

$$\left\{ \begin{array}{r} \sigma\tau | \sigma \in \mathfrak{a},\ \tau \in \mathscr{B}\ \sigma \text{ or } \tau \ \text{ may be empty} \\ \text{but not both.} \end{array} \right\}$$

By 4.1.3 $\mathfrak{a} * \mathscr{B}$ is a simplicial presentation. If $\emptyset$ is empty, we define $\mathfrak{a} * \phi = \phi * \mathfrak{a} = \mathfrak{a}$.

In case $\mathfrak{a}$ and $\mathscr{B}$ are presentations of polyhedra in V and W, then we construct, by 4.1.4, $\widetilde{\mathfrak{a}}$ and $\widetilde{\mathscr{B}}$ which are isomorphic to $\mathfrak{a}$ and $\mathscr{B}$, and for which we can define $\widetilde{\mathfrak{a}} * \widetilde{\mathscr{B}}$. It clearly depends only on $\mathfrak{a}$ and $\mathscr{B}$ upto simplicial isomorphism; in this way we can construct *abstractly* any joins we desire.

Ex. 4.1.10. $\mathfrak{a} * \mathscr{B} = \mathscr{B} * \mathfrak{a}$

$\mathfrak{a} * (\mathscr{B} * \mathscr{C}) = (\mathfrak{a} * \mathscr{B}) * \mathscr{C}$.

That is whenever one side is defined, the other also is defined and both are equal.

Ex. 4.1.11. If $\alpha \in \mathfrak{a}, \beta \in \mathscr{B}$, then

$$Lk(\alpha\beta, \mathfrak{a} * \mathscr{B}) = Lk(\alpha, \mathfrak{a}) * Lk(\beta, \mathscr{B}).$$

In particular, when $\beta = \emptyset$,

$$Lk(\alpha, \mathfrak{a} * \mathscr{B}) = Lk(\alpha, \mathfrak{a}) * \mathscr{B}$$

and when $\mathscr{L} = \emptyset$,

$$Lk(\beta, \mathfrak{a} * \mathscr{B}) = \mathfrak{a} * Lk(\beta, \mathscr{B}).$$

If $\mathfrak{a}$ is the presentation of a single point $\{v\}$, and is joinable to $\mathscr{B}$, then we call $\mathfrak{a} * \mathscr{B}$; the *cone* on $\mathscr{B}$ with *vertex* v, and denote it by $C(\mathscr{B})$. **57**
$\mathscr{B}$ is called the *base* of the cone. If we make the convention, that the unique regular presentation of a one point polyhedron v, is to be written $\{\{v\}\}$, then $C(\mathscr{B}) = \{\{v\}\} * \mathscr{B}$.

Definition 4.1.12. Let $\mathscr{S}$ be a simplicial presentation, and $\sigma \in \mathscr{S}$. Then the *star of* σ *in* $\mathscr{S}$, denoted by $St(\sigma, \mathscr{S})$, is defined to be $\{\overline{\sigma}\} * Lk(\sigma, \mathscr{S})$.

Clearly $St(\sigma, \mathscr{S})$ is a subpresentation of $\mathscr{S}$ and is equal to $\cup\{\{\overline{\tau}\} | \tau \in \mathscr{S}, \sigma \leq \tau\}$.

In case σ contains only a single point x, we write $St(x, \mathscr{S})$.

Ex. 4.1.13. Let $\mathscr{S}$ be a simplicial presentation, σ an element of $\mathscr{S}$. If τ is a face of σ with $\dim \tau = \dim \sigma - 1$, then

$$Lk(\sigma, \mathscr{S}) = Lk(\tau, \{\partial\sigma\} * Lk(\sigma, \mathscr{S})).$$

Definition 4.1.14. If $\mathscr{S}$ is a simplicial presentation, the k-skeleton of $\mathscr{S}$, denoted by $\mathscr{S}_k$ is defined to be

$$\mathscr{S}_k = \cup\{\{\overline{\sigma}\}|\sigma \in \mathscr{S} \dim \sigma \leq k\}.$$

Clearly $\mathscr{S}_k$ is a subpresentation of $\mathscr{S}$.

Ex. 4.1.15. If $\sigma \in \mathscr{S}_k$ and $\dim \sigma = \ell$, $(\ell \leq k)$, then

$$Lk(\sigma, \mathscr{S}_k) = Lk(\sigma, \mathscr{S})_{k-\ell-1}.$$

Ex. 4.1.16. Let $f : P \to Q$ be a polyhedral map, simplicial with respect to presentations $\mathscr{S}$ and $\mathscr{S}'$ of P and Q respectively. Then

(1) $f(\mathscr{S}_k) \subset (\mathscr{S}'_k)$

(2) If $\sigma \in \mathscr{S}$, $f(St(\sigma, \mathscr{S})) \subset St(f\sigma, \mathscr{S}')$

58 (3) For every $\sigma \in \mathscr{S}$, $f(Lk(\sigma, \mathscr{S})) \subset Lk(f(\sigma), \mathscr{S}')$ if and only if f maps every 1-simplex of $\mathscr{S}$ onto a 1-simplex of $\mathscr{S}'$.

(Strictly speaking, these are the maps induced by f).

4.2 Abstract Theory II

Definition 4.2.1. Let $\mathscr{P}$ be a regular presentation and η a centering of $\mathscr{P}$. Let $A \in \mathscr{P}$. Then *the dual of A* and the *link of A, with respect of η*, denoted by δA and λA are defined to be

$$\delta A = \{0(\eta C_0, \ldots, \eta C_k) \mid A \leq C_0 < \ldots < C_k, k \geq 0\}$$
$$\lambda A = \{0(\eta C_0, \ldots, \eta C_k) \mid A < C_0 < \ldots < C_k, k \geq 0\}$$

where $C_i \in \mathscr{P}$ for all i.

Clearly δA and λA are subpresentations of $d\mathscr{P} = d(\mathscr{P}, \eta)$. When there are several regular presentations to be considered, we will denote these by $\delta_{\mathscr{P}} A$ and $\lambda_{\mathscr{P}} A$. η will be usually omitted from the terminology, and these will be simply called *dual of A* and *link of A*.

4.2.2 Every simplex of $d\mathscr{P}$ belongs to some δA.

4.2.3 δA is the cone on λA with vertex ηA.

Ex. 4.2.4. Let $\dim A = p$, and consider any p-simplex σ of $d\mathscr{P}$ contained in A i.e. $\sigma = 0(\eta B_0, \ldots, \eta B_p)$, for some $B_0 < \ldots < B_p = A$. Then $\lambda A = Lk(\sigma, d\mathscr{P})$.

4.2.5 Suppose $\mathscr{P}$ is, in fact, simplicial. Then we have defined both λA and $Lk(A, \mathscr{P})$. These are related thus:

A vertex of λA is of the form ηC where $A < C$. There is a unique **59**
B of $\mathscr{P}$ such that $C = AB$. ηB is a typical vertex of $d(Lk(A, \mathscr{P}))$. The correspondence $\eta C \leftrightarrow \eta B$ defines a simplicial isomorphism:

$$\lambda A \leftrightarrow d(Lk(A, \mathscr{P})).$$

Ex. 4.2.6. With the notation of 4.2.1, $A < B$ if and only if $\delta B \subset \lambda A$. For any $A \in \mathscr{P}$, λA is the union of all δB for $A < B$.

Ex. 4.2.7. If $\mathscr{P}$ is *simplicial*, $A, B \in \mathscr{P}$, then $\delta A \cap \delta B \neq \emptyset$ if and only if A and B are faces of a simplex of $\mathscr{P}$. If C is the minimal simplex of $\mathscr{P}$ containing both A and B (that C is the open simplex generated by the union of the vertices of A and B), then $\delta A \cap \delta B = \delta C$.

Definition 4.2.8. If $\mathscr{P}$ is a regular presentation and η a centering of $\mathscr{P}$, *the dual k-skeleton of $\mathscr{P}$*, denoted by $\mathscr{P}^k$ is defined to be

$$\mathscr{P}^k = \left\{ \begin{array}{r} 0(\eta C_0, \ldots, \eta C_p) \mid C_0 < \ldots < C_p, \dim C_0 \geq k, \\ p \geq 0, \quad C_i \in \mathscr{P}. \end{array} \right\}$$

Clearly $\mathscr{P}^k$ is a subpresentation of $d\mathscr{P}$, and is, in fact the union of all δA for $\dim A \geq k$. It is even the union of all δA for $\dim A = k$.

Thus δA, λA, $\mathscr{P}^k$ are all simplicial presentations.

Ex. 4.2.9. $d\mathscr{P} = \mathscr{P}^0 \supset \mathscr{P}' \supset \ldots \supset \mathscr{P}^n \supset \mathscr{P}^{n+1} = \emptyset$, where n is the dimension of $\mathscr{P}$. Dim $\mathscr{P}^k = n - k$.

We shall be content with the computation of links of vertices of $\mathscr{P}^k$.

60 **Ex. 4.2.10.** If $A \in \mathscr{P}$, $\dim A \geq k$, then

$$Lk(\eta A, \mathscr{P}^k) = \{\partial A\}^k * \lambda A.$$

Next, we consider the behaviour of polyhedral maps with respect to duals.

Let $f : P \to Q$ be a polyhedral map; let $\mathscr{P}$ and $\mathcal{Q}$ be two simplicial presentations of P and Q respectively with respect to which f is simplicial. If $\mathscr{V}$ is a centering of $\mathcal{Q}$, it can be *lifted* to a centering η of $\mathscr{P}$, that is $f(\eta A) = \mathscr{V} f(A)$ for all $A \in \mathscr{P}$. (see 2.2). f is simplicial with respect to $d(\mathscr{P}, \eta)$ and $d(\mathcal{Q}, \mathscr{V})$ also. Now,

4.2.11 If $A \in \mathscr{P}$, $f(\delta_{\mathscr{P}} A) \subset \delta_{\mathcal{Q}}(f(A))$.

4.2.12 If $B \in \mathcal{Q}$, then $f^{-1}(\delta_{\mathcal{Q}} B) = \cup\{\delta_{\mathscr{P}} A | f(A) = B\}$.

Remark . All these should be read as maps induced by f, etc. Since each such A must have dimension $\geq \dim B$, we have

Proposition 4.2.13. *With the above notation, for each k,* $f^{-1}(\mathcal{Q}^k) \subset \mathscr{P}^k$.

This property is dual to the property with respect to the usual skeleta "$f(\mathscr{P}_k) \subset \mathcal{Q}_k$".

Corollary 4.2.14. *If* $\dim P = n$, *then* $\dim f^{-1}(\mathcal{Q}^k) \leq n - k$.

In particular, if $\dim Q = m$, and q is a point of an (open) m - dimensional simplex of $\mathcal{Q}$, $f^{-1}(q)$ is $a \leq (n - m)$-dimensional subpolyhedron of P.

Ex. 4.2.15. $f^{-1}(\mathcal{Q}') = \mathscr{P}^1$, if and only if every 1-simplex of $\mathscr{P}$ is mapped onto a 1-simplex of $\mathcal{Q}$. (i.e. no 1-simplex of $\mathscr{P}$ is collapsed to a simple point).

4.3 Geometric Theory

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Definition 4.3.1. Let P and Q be polyhedra in the same vector space V. We say that the *join of P and Q is defined* (or $\underline{P * Q}$ *is defined*, of *P and Q are joinable*), if:

(a) $P \cap Q = \emptyset$

(b) If $x, x' \in P$, $y, y' \in Q$, and either $x \neq x'$ or $y \neq y'$; then $O(x, y) \cap O(x', y') \neq \emptyset$.

If the join of P and Q is defined, we define the *join* of P and Q, denoted by $P * Q$ to be

$$P * Q = \cup\{x, y] \mid x \in P, y \in Q\}$$

By definition, $P * \emptyset = \emptyset * P = P$.

Every point $z \in P * Q$ can be written as:

$$z = (1 - t)x + ty, \quad x \in P, \quad y \in Q, \quad 0 \leq t \leq 1.$$

The number t is uniquely determined by z; y is uniquely determined if $z \notin P$ (i.e. if $t \neq 0$), x is uniquely determined if $z \notin Q$ (i.e. if $t \neq 1$).

4.3.2 Let $\mathscr{P}$ and $\mathcal{Q}$ be simplicial presentations of P and Q; and suppose the (geometric) join $P*Q$ is defined. Then by 4.1.2, the (simplicial) join $\mathscr{P} * \mathcal{Q}$ is defined, and we have $|\mathscr{P} * \mathcal{Q}| = P * Q$.

This shows that $P * Q$ is a polyhedron.

Definition 4.3.3. If P_1, Q_1, P_2, Q_2 are polyhedra such that $P_1 * Q_1$ and $P_2 * Q_2$ are defined, and $f : P_1 \to P_2$, $g : Q_1 \to Q_2$ are maps, *then the join of f and g*, denoted by $f * g$, is the map from $P_1 * Q_1$ to $P_2 * Q_2$ given by,

$$(f * g)((1 - t)x + ty) = (1 - t)f(x) + tg(y)$$

$x \in P_1$, $y \in Q_1$, $0 \leq t \leq 1$.

4.3.4 In the above if $f : \mathcal{P}_1 \to \mathcal{P}_2$, $g : \mathcal{Q}_1 \to \mathcal{Q}_2$ are simplicial with **62**
respect to $\mathscr{P}_1$, $\mathscr{P}_2$; $\mathcal{Q}$, $\mathcal{Q}_2$, then $f * g : P_1 * Q_1 \to P_2 * Q_2$ is simplicial with respect to $\mathscr{P}_1 * \mathcal{Q}_1$ and $\mathscr{P}_2 * \mathcal{Q}_2$. Thus the join of polyhedral maps is polyhedral.

4.3.5 If $P * Q$ is defined, $(\mathrm{Id})_p * (\mathrm{Id}_Q) = \mathrm{Id}_{P*Q}$. If, $P_1 * Q_1$, $P_2 * Q_2$, $P_3 * Q_3$ are defined and $f_1 : P_1 \to P_2$, $f_2 : P_2 \to P_3$, $g_1 : Q_1 \to Q_2$, $g_2 : Q_2 \to Q_3$ are maps, then

$$(f_2 \circ f_1) * (g_2 \circ g_1) = (f_2 * g_2) \circ (f_1 * g_1)$$

This says that the join is a functor of two variables from pairs of polyhedra for which join is defined and pairs of polyhedral maps, to polyhedra and polyhedral maps.

The join of a polyhedron P and a single point v is called the *cone* on P, (sometimes denoted by $C(P)$) with *base* P and vertex v.

Ex. 4.3.6. $C(P)$ is contractible.

Ex. 4.3.7. $P * Q - Q$ contains P as a deformation retract.

Hint: Use the map given by $(*)$ below.

Let us suppose that $P * Q$ and the cone $C(Q)$ with vertex v are both defined. The interval $[0, 1]$ is $0 * 1$, and so two maps can be defined:

$$\begin{aligned} &\beta : P * Q \to [0, 1], \quad \text{the join of} \quad P \to 0, Q \to 1, \\ &\alpha : C(Q) \to [0, 1], \quad \text{the join of} \quad v \to 0, Q \to 1. \end{aligned}$$

Simply speaking,

$$\begin{aligned} &\alpha((1 - t)x + ty) = t \\ &\beta((1 - t)v + ty) = t, \quad \text{for} \quad x \in P, y \in Q. \end{aligned}$$

63 The correspondence:

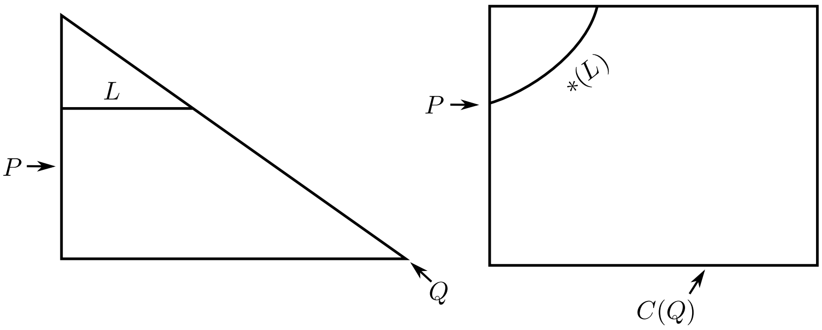
$$(*) \qquad (1 - t)x + ty \leftrightarrow (x, (1 - t)v + ty)$$

is a well defined function between

$$\alpha^{-1}([0.1)) \quad \text{and} \quad P \times \beta^{-1}([0, 1)).$$

It is a homeomorphism, in fact. But it fails to be in any sense polyhedral, since it maps, in general, line segments into curved lines.

Example. Taking P to be an interval, Q to be a point.



The horizontal line segment corresponds to the part of a hyperbola under the above correspondence.

We can however find a polyhedral substitute for this homeomorphism.

Proposition 4.3.8. *Let P, Q, α, β be as above, let* $0 < \tau < 1$. *Then there is a polyhedral equivalence.*

$$\mathcal{L}^{-1}([0,\tau]) \approx P \times \beta^{-1}([0,\tau])$$

which is consistant with the projection onto the interval $[0,\tau]$.

Proof. Let $\mathscr{P}$ and $\mathcal{Q}$ be simplicial presentations of P and Q and take the simplicial presentation $\mathscr{T} = \{\{0\},\{\tau\},(0,\tau)\}$ of $[0,\tau]$.

Consider the set of all sets of the form $A(\rho,\sigma,i)$, where $\rho \in \mathscr{P}$, **64**
$\sigma \in \mathcal{Q}$, $i \in \mathscr{T}$ and $\sigma = \emptyset$ iff $i = \{0\}$, defined thus:

$$A(\rho,\emptyset,0) = \rho$$
$$A(\rho,\sigma,i) = \rho\sigma \cap \mathcal{L}^{-1}(i)$$

The set of all these $A(\rho,\sigma,i)$, call it $\mathfrak{a}$. It is claimed that $\mathfrak{a}$ is a regular presentation of $\mathcal{L}^{-1}([0,\tau])$, and that $A(\rho,\sigma,i) \leq A(\rho',\sigma',i')$ if and only if $\rho \leq \rho'$, $\sigma \leq \sigma'$, $i \leq i'$.

Secondly, consider the set of all sets of the form $B(\rho, \sigma, i)$ where $\rho \in$ *mathscrP*, $\sigma \in \mathcal{Q}$, $i \in \mathscr{T}$, and $\sigma = \emptyset$ if $i = \{0\}$ defined thus:

$$B(\rho, \emptyset, 0) = \rho \times \{v\}$$
$$B(\rho, \sigma, i) = \rho \times (\sigma\{v\} \cap \beta^{-1}(i))$$

It is claimed that $\mathscr{B}$ of all such $B(\rho, \sigma, i)$ is a regular presentation of $P \times \mathscr{B}^{-1}([0, \tau])$, and that $B(\rho, \sigma, i) \leq B(\rho', \sigma', i')$ if and only if $\sigma \leq \sigma'$, $\rho \leq \rho'$ and $i \leq i'$.

Hence the correspondence $A(\rho, \sigma, i) \leftrightarrow B(\rho, \sigma, i)$ is a combinatorial equivalence $\mathfrak{a} \leftrightarrow \mathscr{B}$. If we choose the centerings η and $\mathscr{V}$ of $\mathfrak{a}$ and $\mathscr{B}$ respectively such that

$$\mathcal{L}(\eta(A(\rho, \sigma, (0, \tau))) = \tau/2$$

and β (2 nd coordinate of $\mathscr{V}(B(\rho, \sigma, (0, \tau))) = \tau/2$. The induced simplicial isomorphism $d(\mathfrak{a}, \eta) \leftrightarrow d(\mathscr{B}, \mathscr{V})$ gives a polyhedral equivalence $\mathcal{L}^{-1}([0, \tau)] \approx P \times \mathscr{B}^{-1}([0, \tau])$, consistent with the projection onto $[0, \tau]$.

65 It should perhaps be remarked that by choosing $\mathscr{P}$ and $\mathcal{Q}$ fine enough, our equivalence is arbitrarily close to the correspondence (*) on page 67. □

Corollary 4.3.9. *Let $C(P)$ be the cone on P with vertex v, and $\mathcal{L}$: $C(P) \to [0, 1]$ be the join of $P \to 0$, $v \to 1$. Then for any $\tau \in (0, 1)$, $\mathcal{L}^{-1}([0, \tau])$ is polyhedrally equivalent to $P \times [0, \tau]$ by an equivalence consistent with the projection to $[0, \tau]$.*

For, take $Q = v$ in 4.3.8.

Corollary 4.3.10. *Let $\mathcal{L} : P * Q \to [0, 1]$ be the join of $P \to 0$, $Q \to 1$; let $0 < \gamma < \delta < 1$. Then $\mathcal{L}^{-1}([\gamma, \delta])$ is polyhedrally equivalent to $P \times Q \times [\gamma, \delta]$ by an equivalence consistent with the projection to $[\gamma, \delta]$.*

For, by 4.3.8, $\mathcal{L}^{-1}([0, \delta]) \approx P \times \beta^{-1}([0, \delta])$ where $\beta : C(Q) \to [0, 1]$ is the join of $Q \to 1$ and vertex $\to 0$. By 4.3.9, interchanging 0 and 1, we see that $\beta^{-1}([\gamma, 1] \approx Q \times [\gamma, 1]$; combining these and noting the preservation of projection on $[\gamma, \delta]$, we have the desired result.

Definition 4.3.11. Let K be a polyhedron and $x \in K$. Then a subpolyhedron L of K is a said to be a (polyhedral) *link* of x in K, if $L * x$ is defined, is contained in K, and is a neighbourhood of x in K.

A (polyhedral) *star* of x in K is the cone with vertex x on any link of x in K.

Clearly, if $a \in K_1 \subset K$, and K_1 is a neighbourhood of 'a' in K, then $L \subset K_1$ is a link of 'a' in K_1, if and only if it is a link of 'a' in K.

To show that links and stars exist, we triangulate K by a simplicial presentation $\mathscr{S}$ with x as a vertex. Then $|Lk(x, \mathscr{S})|$ is a link of x in K, and $|St(x, \mathscr{S})|$ is a star of x in K. In this case $|St(x, \mathscr{S})| - |Lk(x, \mathscr{S})|$ is open in K; this need not be true for general links and stars. **66**

Ex. 4.3.12. If $\mathscr{S}$ is any simplicial presentation of K, and $x \in \sigma \in \mathscr{S}$, then $|\{\partial\sigma\} * Lk(\sigma, \mathscr{S})|$ is a link of x in K, and $|\{\overline{\sigma}\} * Lk(\sigma, \mathscr{S})|$ is a star of x in K.

(b) With δA, λA as in 4.2.1, if $x \in A$, $\partial A * |\lambda A|$ is a link of x in K.

Ex. 4.3.12′. (a) Let $f : K \to K'$ be a one-to-one polyhedral map, simplicial with reference to presentations $\mathscr{S}$ and $\mathscr{S}'$ of K and K'. Then for any $\sigma \in \mathscr{S}$, $x \in \sigma$

$$f||\{\overline{\sigma}\} * Lk(\sigma, \mathscr{S})| \quad \text{is the join of}$$
$$f||\{\partial\sigma\} * Lk(\sigma, \mathscr{S})| \quad \text{and } x \to f(x).$$

Formulate and prove a more general statement using 4.1.16

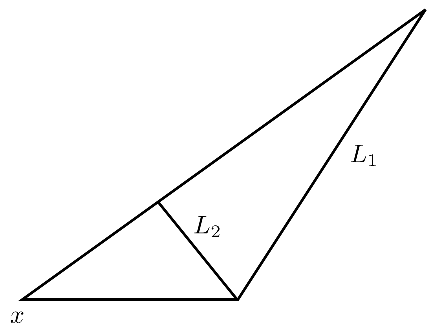
(b) With the hypothesis of 4.2.15, if A_0 is a 0-cell of $\mathscr{P}$, $f(|\lambda A_0|) \subset |\lambda(fA_0)|$ and

$$f||\delta A_0| \quad \text{is the join of} \quad A_0 \to f(A_0) \quad \text{and} \quad f||\lambda A_0|.$$

If x and a are two distinct points in a vector space, the set of points $(1-t)x + ta$, $t > 0$ will be called '*the ray from x through a*'.

Let L_1 and L_2 be two links of x in k, then for each point $a \in L_1$, the ray through a from x intersects L_2 in a unique point $h(a)$ (and every point in L_2 is such a image). It intersects L_2 in atmost one point, since **67** the cone on L_2 with vertex x exists. It intersects L_2 in at least one point since the cone on L_2 must contain a neighbourhood of the vertex of the cone on L_1.

The function $h : L_1 \to L_2$ thus defined is a homeomorphism. But, perhaps contrary to intution, it is *not* polyhedral.



The graph of the map h in this simple case is a segement of a hyperbola.

The fallacy of believing h is polyhedral is old (See, Alexander "The combinatorial theory of complexes", Annals of Mathematics, 31, 1930); for this reason we shall call h the *standard mistake* after Zeeman (see Chapter I of "Seminar on Combinatorial Topology"). We shall show how to approximate it very well by polyhedral equivalences.

It might be remarked that the standard mistake is "piecewise projective", the category of such maps has been studies by N.H. Kuiper [see "on the Smoothings of Triangulated and combinatorial Manifolds" in "Differential and combinatorial Topology", A symposium in Honor of Marston Morse, Edited by S.S. Cairns].

Definition 4.3.13. Let A and B be two convex sets. A one-to-one function from A onto B, $\mathcal{L} : A \to B$ is said to be *quasi-linear*; if for each a_1, $a_2 \in A$, $\mathcal{L}([a_1, a_2]) = [\mathcal{L}(a_1), \mathcal{L}(a_2)]$.

68 In other words, $\mathcal{L}$ preserves line segments. It is easy to see that $\mathcal{L}^{-1}$ is also quasi-linear.

Example. Any homeomorphism of an interval is quasi-linear. In $\mathbb{R}^2$, the map:

$$(r_1, r_2) \to \left(\frac{r_1}{1 - r_1}, \frac{r_2}{1 - r_1} \right)$$

as a map from $A = \{(r_1, r_2) \mid 0 < r_i < 1\}$ to $B = \{(\mathscr{S}_1, \mathscr{S}_2) \mid \mathscr{D} > \mathscr{S}_i > 0\}$ is quasi-linear.

Proposition 4.3.14. *Let $\mathcal{L} : A \to B$ be quasi-linear. Let $\{a_0, \ldots, a_n\}$ be an independent set of points in A, defining an open simplex σ. Then $\{\mathcal{L}(a_0), \ldots, \mathcal{L}(a_n)\}$ is independent, and the simplex they define is $\mathcal{L}(\sigma)$. Consequently τ is a face of σ if and only if $\mathcal{L}(\tau)$ is a face of $\mathcal{L}(\sigma)$.*

The proof is by induction. For $n = 1$, this is the definition. The inductive step follows by writing $\sigma = \sigma'\{a_n\}$ and noting that quasi-linear map preserves joins.

Theorem 4.3.15. *Let L_1 and L_2 be two links of x in K with $h : L_1 \to L_2$, the standard mistake. Suppose $\mathscr{Z}_1$ and $\mathscr{Z}_2$ are polyhedral presentations of L_1 and L_2. Then there exist simplicial refinements $\mathscr{S}_1$ and $\mathscr{S}_2$ of $\mathscr{Z}_1$ and $\mathscr{Z}_2$ such that for each $\sigma \in \mathscr{S}_1$, $h(\sigma) \in \mathscr{S}_2$ and $h|\overline{\sigma}$ is quasi-linear. If $f : L_1 \to L_2$ is defined as the linear extension of h restricted to the vertices of $\mathscr{S}_1$, then f is a polyhedral equivalence simplicial with respect to $\mathscr{S}_1$ and $\mathscr{S}_2$ and such that $f(\sigma) = h(\sigma)$ for all $\sigma \in \mathscr{S}$.*

Proof. We can suppose that $\mathscr{Z}_1$ and $\mathscr{Z}_2$ are simplicial, and find a simpli- **69**
cial presentation $\mathscr{P}$ of $(L_1 * x \cup L_2 * x)$ refining $(\mathscr{Z}_1 * \{\{x\}\} \cup \mathscr{Z}_2 * \{\{x\}\})$. Define $\mathscr{S} = Lk(x, \mathscr{P})$. It is clear that every simplex $\sigma \in \mathscr{S}$ is contained in $\tau * \{x\}$, for $\tau \in \mathscr{Z}_1$, and hence the standard mistake $h_1 : |\mathscr{S}| \to L_1$ takes $\overline{\sigma}$ to $h_1(\overline{\sigma}) \subset \overline{\tau}$.

The restriction of h_1 to $\overline{\sigma}$ is quasi-linear. For, let $a_1, a_2 \in \overline{\sigma}$; the three points a_1, a_2, x determine a plane and in that plane an angular region L, which is the union of all rays from x through the points of $[a_1, a_2]$. The standard mistake, by definition, takes $[a_1, a_2] \subset \overline{\sigma}$ to $L \cap \overline{\tau}$, which, it is geometrically obvious, is just $[h_1(a_1), h_1(a_2)]$.

This, together with 4.3.14, enables us to define $\mathscr{S}_1 = \{h_1(\sigma) \mid \sigma \in \mathscr{S}\}$, and to see that this is a simplicial presentation refining $\mathscr{Z}_1$.

Similarly, via the standard mistake $h_2 : |\mathscr{S}| \to L_2$, we have

$$\mathscr{S}_2 : \{h_2(\sigma) \mid \sigma \in \mathscr{S}\}.$$

Since, clearly, $h : L_1 \to L_2$ is $h_2 \circ h_1^{-1}$ and the composition and inverse of quasi-linear maps are again quasi-linear, the major part of the theorem is proved.

The last remark about f is obvious. □

4.3.16 If in 4.3.15, for a subpolyhedron K' of L_1, $h|K'$ is polyhedral, then we can arrange for $f : L_1 \to L_2$ of the theorem to be such that $f|K' = h|K'$.

70 For, all we need to do is to assure that $\mathscr{Z}_1$ has a subpresentation covering K; then because h is linear on each simplex in K, the resultant f is identical with h there.

Corollary 4.3.17. *Links (resp. stars) of x in K exist and all are all polyhedrally equivalent.*

Proposition 4.3.18. *If $f : P \to Q$ is a polyhedral equivalence, then any link of x in P is polyhedrally equivalent to any link of x in Q.*

For triangulate f, and look at the simplicial links; they are obviously isomorphic.

4.3.18[1]. Allows to define the *local dimension* of *polyhedron K at x.* This is defined to be the dimension of any star of x in K. By 4.3.17 this is well defined. It can be easily seen that (by 4.3.12) the closure of the set of points where the local dimension is p is a subpolyhedron of K, for any integer p.

We will next consider links and stars in products and joins.

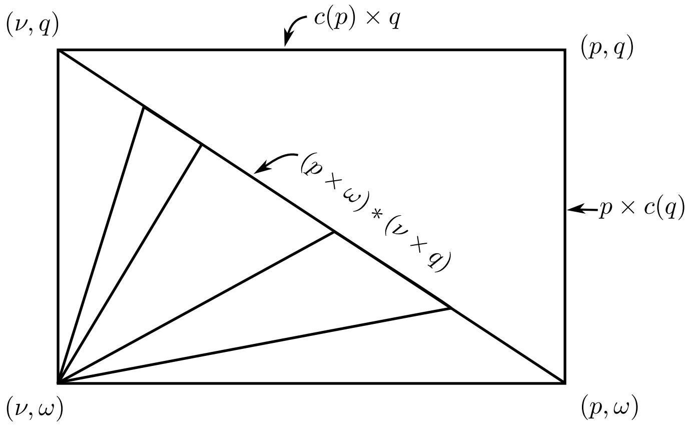
Ex. 4.3.19. Let $C(P)$ and $C(Q)$ be cones with vertices v and w. Let $Z = (P \times C(Q)) \cup (C(P) \times Q)$. Then

(a) $C(P) \times C(Q) = C(Z)$, the cone on Z with vertex (v, w)

(b) $P \times w$ and $v \times Q$ are joinable, and $(P \times w) * (v \times Q)$ is a link of (v, w) in $C(Z)$.

Hence by straightening our the standard mistake, we get a polyhedral equivalence $P*Q \approx Z$, which extends the canonical maps $P \to P\times w$ and $Q \to v \times Q$. **71**

Hint: It is enough to look at the following 2-dimensional picture for arbitrary $p \in P$, $q \in Q$:



Ex. 4.3.20. Prove that $P * Q \approx (C(p) \times Q \cup P \times C(Q))$ utilising 4.3.8. If $\varphi : P * Q \to [0, 1]$ is the join of $P \to 0$, $Q \to 1$, the equivalence can be chosen so that $\varphi^{-1}([0, 1/2])$ goes to $P \times C(Q)$ and $\varphi^{-1}([1/2, 1])$ goes to $C(P) \times Q$.

Ex. 4.3.21 (Links in products)**.** If $x \in P$, $y \in Q$, then a link of (x, y) in $P \times Q$ is the join of a link of x in P and a link of y in Q.

The join of X to a polyhedron $\{x_1, x_2\}$ consisting of two points is called *the suspension* of X with *vertices* x_1 and x_2 and is denoted by $S(X)$. Similarly K^{th} order suspensions are defined.

Ex. 4.3.22 (Links in joins)**.** In $P * Q$.

(1) Let $x \in P * Q - (P \cup Q)$, and let $x = (1 - t)p + tq$, $p \in P$, $q \in Q$, $0 < t < 1$. If L_1 is a link of p in P, L_2 a link of q in Q, then $S(L_1 * L_2)$ (with vertices p, q) is a link of x in $P * Q$. **72**

(2) If $p \in P$, and L is a link of p in P, then $L * Q$ is a link of p in $P * Q$.

Hint: for 1. Consider simplicial presentations $\mathscr{P}$ and $\mathcal{Q}$ of P and Q having p and q as vertices. Then $Lk(p, \mathscr{P}) * Lk(q, \mathcal{Q})$ is a link of $O(p, q)$ in $\mathscr{P} * \mathcal{Q}$, by 4.1.11. Hence a link of x in $P * Q = |\{p, q\} * Lk(p, \mathscr{P}) * Lk(q, \mathcal{Q})|$ or the suspension of $Lk(p, \mathscr{P}) * Lk(q, \mathcal{Q})$ with vertices p and q. The general case follows from this.

4.4 Polyhedral cells, spheres and Manifolds

In this section, we utilize links and stars to define polyhedral cells, spheres and manifolds and discuss their elementary properties.

Let us go back to the open and closed (convex) cells discussed in 1.5. If A is an open cell, then the closed cell $\overline{A}$ is the cone over ∂A with vertex a, for any $a \in A$.

Proposition 4.4.1. *If A and B are two open cells of the same dimension, then ∂A and ∂B are polyhedrally equivalent. Moreover the equivalence can be chosen to map any given point x of ∂A onto any given point y of ∂B.*

Proof. Let $\dim A = n = \dim B = n$. Via, a linear isomorphism of the
73 linear manifolds containing A and B, we can assume that A and B are in the same n-dimensional linear manifold, and moreover that $A \cap B \neq \emptyset$. Then ∂A and ∂B are both links of any point of $A \cap B$ in $\overline{A} \cup \overline{B}$. Hence ∂A and ∂B are polyhedrally equivalent. A rotation of A will arrange for the standard mistake to map x to y. And 4.3.15, we can clearly arrange for x and y to be vertices in $\mathscr{S}_1$ and $\mathscr{S}_2$. □

By joining the above map with a map of point of A to a point of B, we can extend it to a polyhedral equivalence of $\overline{A}$ and $\overline{B}$. Thus any two closed cells are polyhedrally equivalent.

Definition 4.4.2. A *polyhedral n-sphere* (or briefly an *n-sphere*) is any polyhedron, polyhedrally equivalent to the boundary of an open cell of dimension $(n + 1)$.

By 4.4.1 this is well defined.

Definition 4.4.3. A *polyhedral n-cell* (or briefly an *n-cell*) is any polyhedron, polyhedrally equivalent to a closed convex cell of dimension n.

By the remark after 4.4.1, this is well defined. All the cells and spheres except the 0-sphere are connected.

Consider the "standard n-cell", the closed n-simplex, and the "standard $(n-1)$-sphere", the boundary of a n-simplex. By 4.1.8, 4.3.18 and 4.4.1, we have

Proposition 4.4.4. *The link of any point in an n-sphere is an $(n-1)$-sphere.*

Corollary 4.4.5. *An n-sphere is not polyhedrally equivalent to an (m)-sphere, if $m \neq n$.*

Proof. By looking at the links using 4.4.4, and induction. □ **74**

4.4.6 If $f : D \approx \overline{\sigma}$ is an equivalence of an n-cell with a closed n-simplex $\overline{\sigma}$, we see that for points of C corresponding to points of $\partial\sigma$, the link in C is an $(n-1)$-cell; and for points of C corresponding to points of σ, the link in C is an $(n-1)$-sphere.

Proposition 4.4.7. *An n-sphere is not polyhedrally equivalent to an n-cell.*

Proof. Again by induction. For $n = 0$, a sphere has two points and a cell has only one point.

For $n > 0$, an n-cell has points which have $(n-1)$-cells as links, where as in a sphere all points have $(n-1)$-spheres as links. And so, by induction on n they are different. □

This allows us to define boundary for arbitrary n-cells, namely the boundary of an n-cell C, is the set of all points of C whose links are $(n-1)$-cells. We will denote this also by ∂C. This coincides with the earlier definition for the boundary of a closed convex cell, and the boundary of a n-cell is an $(n-1)$-sphere. And as in 4.4.5, an n-cell and an (m)-cell are not polyhedrally equivalent if $m \neq n$.

By taking a particularly convenient pairs of cells and sphere, the following proposition is easily proved:

Ex. 4.4.8. Whenever they are defined,

(1) The join a m-cell and an n-cell is a $(m+n+1)$-cell.

(2) The join of a m-cell and an n-sphere is a $(m+n+1)$-cell.

75 (3) The join of a m-sphere and an n-sphere is a $(m+n+1)$-sphere.

If in (1) of 4.4.8, C_1 and C_2 are the cells, then $\partial(C_1 * C_2) = \partial C_1 * C_2 \cup C_1 * \partial C_2$. In (2) of 4.4.8, if C is the cell, and S the sphere $\partial(C * S) = \partial C * S$.

Definition 4.4.9. A *PL-manifold of dimension n* (or a *PL n*-manifold) is a polyhedron M such that for all points $x \in M$, the link of x in M is either an $(n-1)$-cell or an $(n-1)$-sphere.

Definition 4.4.10. If M is a PL n-manifold, then the boundary of M denoted by ∂M, defined to be $\partial M = \{x \in M \mid \text{ link of } x \text{ in } M \text{ is a cell}\}$.

Notation. If A is any subset of M, the interior of A and the boundary of A in the topology of M, will be denoted by int $_M A$ and bd $_M A$ respectively. $M - \partial M$ is also usually called the interior of M, this we will denote by int M or $\mathring{M}$. Note that $\text{int}_M M = M$, where as int $M = M - \partial M$.

It is clear from the proposition above, the manifolds of different dimensions cannot be polyhedrally equivalent, of course, from Brouwer's theorem on the "Invariance of domain", it follows that they cannot even be homeomorphic.

Proposition 4.4.11. *If M is a PL n-manifold, then ∂M is a PL $(n-1)$-manifold, and $\partial(\partial M) = \emptyset$.*

Proof. We first observe that $M - \partial M$ is open in M. For if $x \in M - \partial M$, let L be a link of x in M, S the corresponding star, such that $S - L$ is
76 open in M. S is a cell and $\partial S = L$. If $y \in S - L$, then a link of y in S is

a link of y in M, since S is a neighbourhood of y. Since S is a cell and $y \in S - \partial S$, the link of y in S is a sphere. Hence $y \in M - \partial M$, for all $y \in S - L$ or $M - \partial M$ is open in M. Hence ∂M is closed in M.

If $\mathscr{S}$ is any simplicial presentation of M and $\sigma \in \mathscr{S}$, the $|\{\partial\sigma\} * Lk(\sigma, \mathscr{S})\}$ is a link of x in M for all $x \in \sigma$ by 4.3.12. Hence $\sigma \subset \partial M$ or $\sigma \subset M - \partial M$. If $\sigma \subset \partial M$, $\partial\sigma$ also is contained in ∂M, since ∂M is closed. ∂M being the union of all such $\overline{\sigma}$ is a subpolyhedron of M.

Let x be a point of ∂M, L a link of x in M and $S = L * x$, the corresponding star such that $S - L$ is open. L is an $(n-1)$-cell. And by 4.4.8, S is an n-cell with $\partial S = L \cup x * \partial L$. If $y \in x * \partial L - \partial L \subset S - L$, then a link of y in S is a link of y in M as above. But a link of y in S is a cell, since $y \in \partial S$. Hence $x * \partial L - \partial L \subset \partial M$. Since ∂M is closed, $x * \partial L \subset \partial M$, and since $x * \partial L$ is a neighbourhood of x in ∂M, ∂L is a link of x in ∂M. Hence ∂M is a PL $(n-1)$-manifold without boundary. □

Remark. Thus, if $x \in \partial M$, there exist links L of x in M (for example, the links obtained using simplicial presentations), such that $\partial L \subset \partial M$ and ∂L a link of x in ∂M. This need notbe true for arbitrary links. Also there exists links L of $x \in \partial M$ in M, such that $L \cap \partial M = \partial L$. For example, take a regular presentation $\mathscr{P}$ of M in which x is a vertex and take $|\delta_{\mathscr{P}}\{x\}|$.

Proposition 4.4.12. *Let M be a PL n-manifold, and $\mathscr{S}$ a simplicial presentation of M. If $\sigma \in \mathscr{S}$, then either $\sigma \subset \partial M$ or $M - \partial M$, and* 77

(1) *$|Lk(\sigma, \mathscr{S})|$ is a $(n-k-1)$-cell if $\sigma \subset \partial M$*

(2) *$|Lk(\sigma, \mathscr{S})|$ is a $(n-k-1)$-sphere if $\sigma \subset M - \partial M$ where k is the dimension of σ.*

Proof. That $\sigma \subset \partial M$ or $M - \partial M$ is proved in 4.4.11. The proof of (1) and (2) is by induction on k. It $k = 0$, this follows from definition. If $k > 0$, let τ be a $(k-1)$-face of σ. Then $Lk(\sigma, \mathscr{S}) = Lk(\tau, \{\partial\sigma\} * Lk(\sigma, \mathscr{S}))$, and $|\{\partial\sigma\} * Lk(\sigma, \mathscr{S})|$ being the link of a point in σ is either $(n-1)$-sphere or a $(n-1)$-cell. Hence, by induction, $Lk(\sigma, \mathscr{S})$ is either a cell or sphere of dimension $(n-1) - (k-1) - 1 = (n-k-1)$. If $\sigma \subset \partial M$, $|\{\partial\sigma\} * Lk(\sigma, \mathscr{S})|$ is a cell. Hence $|Lk(\sigma, \mathscr{S})|$ cannot be a sphere, since then $|\{\partial\sigma\}| * Lk(\sigma,)| = \partial\sigma * |Lk(\sigma,)|$ would be a sphere. Thus if $\sigma \subset \partial M$,

$|Lk(\sigma, \mathscr{S})|$ is a $(n-k-1)$-cell. Similarly if $\sigma \subset M - \partial M$, $|Lk(\sigma, \mathscr{S})|$ is a $(n-k-1)$-sphere. □

Ex. 4.4.13. (1) Let M be a PL m-manifold, and N a PL n-manifold. Then $M \times N$ is a PL $(m+n)$-manifold and $\partial(M \times N)$ is the union of $\partial M \times N$ and $M \times \partial N$.

Hint: Use, 4.3.21 and 4.4.8.

(2) If $M * N$ is defined, it is not a manifold except for the three cases of 4.4.8.

Hint: Use 4.3.22.

78 **Proposition 4.4.14.** (a) *If $f : S \to S'$ is a one-to-one polyhedral map of an n-sphere S into another n-sphere S', then f is onto.*

(b) *If $f : C \to C'$ is a one-to-one polyhedral map of an n-cell C into another n-cell C' such that $f(\partial C) \subset \partial C'$, then f is onto.*

Proof of (a): By induction. If $n = 0$, S has two points and the proposition is trivial. Let $n > 0$. Let f be simplicial with respect to presentations $\mathscr{S}_1$ and $\mathscr{S}_2$ of S and S'. If x is any point of S, $x \in \sigma$ for some $\sigma \in \mathscr{S}_1$. Consider

$$L_1 = |\{\partial\sigma\} * Lk(\sigma, \mathscr{S}_1)|,\ S_1 = |\{\partial\overline{\sigma}\} * Lk(\sigma, \mathscr{S}_1)|,$$
$$L_2 = |\{\partial(f\sigma)\} * Lk(f\sigma, \mathscr{S}_2)|, \text{ and } S_2 = |\{f\overline{\sigma}\} * Lk(f\sigma, \mathscr{S}_2)|.$$

Since f is injective f maps $L_1 \to L_2$, and $f|S_1$ is the join of $f|L_1$ and $x \to f(x)$. L_1 and L_2 are $(n-1)$-spheres, and by induction $f|L_1$ is bijective. Therefore $f(S_1) = S_2$. Hence $f(S)$ is open in S'. Since S is compact $f(S)$ is closed in S'. Since S is connected, $f(S) = S'$. (b) is proved similarly.

By the same method, it can be shown

Ex. 4.4.15. There is no one-to-one polyhedral map of an n-sphere into an n-cell.

Ex. 4.4.16. (1) A PL-manifold cannot be imbedded in another PL-manifold of lower dimension.

(2) If M and N are two connected manifolds of the same dimension, $\partial N \neq \emptyset$, and $\partial M = \emptyset$, then M cannot be embedded in N. If ∂N is also empty, and if M can be embedded in N then $M \approx N$.

Ex. 4.4.17. (1) If $M \subset N$ are two PL n-manifolds, then $M - \partial M \subset$ **79**
$N - \partial N$, and $M - \partial M$ is open in N. **Hint:** Use 4.4.14 and 4.4.15.

In particular any polyhedral equivalence of N has to take $N - \partial N$ onto $N - \partial N$ and ∂N onto ∂N.

(2) If $M \subset N - \partial N$, both M and N, PL (n)-manifolds, and x any point of ∂M, show that there exist links L of x in N, such that a link of x in M is an $(n-1)$-cell $D \subset L$, and $D \cap \partial M = \partial D$.

Ex. 4.4.18. In 4.2.14, show that if P is a PL n-manifold $f^{-1}(q)(\partial \neq \emptyset)$ is a PL $(n-m)$-manifold and $\partial(f^{-1}(q)) \subset \partial P$.

4.5 Recalling Homotopy Facts

Here we discuss some of the homotopy facts needed later. The reader is referred to any standard book on homotopy theory for the proof of these.

4.5.1 We define a space P to be $(k-1)$-*connected* iff, for any polyhedra $Y \subset X$, with $\dim X \leq k$, every continuous map $Y \to P$ has an extension to X.

Thus, a (-1)-connected polyhedron must just be non-empty. A k-connected polyhedron for $k \leq -2$, can be anything. For $k \geq 0$, it is necessary and sufficient that P be non-empty and that $\pi_i(P) = 0$ for $i \leq k$.

4.5.2 A pair of spaces (A, B) where $B \subset A$, is k-*connected* if for any polyhedra $Y \subset X$ with $\dim X \leq k$, and $f : X \to A$ such that $f(Y) \subset B$, then f is homotopic to a map g, leaving Y fixed, such that $g(X) \subset B$.

This is just the same as requiring that $\pi_i(A, B) = 0$ for $i \leq k$. If A is **80**
contractible (or just $(k-1)$-connected) and (A, B) is k-connected, then B is $(k-1)$-connected.

We shall have occasion to look at pairs of the form $(A, A - B)$, which we denoted briefly as $(A, -B)$. The following discussion is designed to suggest how to prove a result on the connectivity of joins, which is well known from homotopy theory.

4.5.3 Let $A_1 \subset A$, $B_1 \subset B$, and suppose $(A, -A_1)$ is a-connected, $(B, -B_1)$ is b-connected. Then $(A \times B, -A_1 \times B_1)$ is $(a+b+1)$-connected.

Let $Y \subset X$, $\dim X \le a + b + 1$, and $f : X \to A \times B$, with $f(Y) \cap A_1 \times B_1 = \emptyset$.

We must now triangulate X finely by say $\mathscr{S}$. Look at $|\mathscr{S}_a| = X_1$ and $|\mathscr{S}^{a+1}| = X_2$. Then $\dim X_1 \le a$, $\dim X_2 \le b$, and so the two coordinates of f are homotopic, using homotopy extension, to get a map, still called f_1 such that

$$f_A(X_1) \cap A_1 = \emptyset, \quad f_B(X_2) \cap B_1 = \emptyset.$$

Because $X - X_2$ has X_1 as a deformation retract, we can first get $f_B(X_2) \cap B_1 = \emptyset$ and then $f^{-1}(A_1 \times B_1)$ is contained in $X - X_2$. By changing, homotopically, only the first coordinate, we get $f^{-1}(A_1 \times B_1) = \emptyset$.

To go more deeply into this sort of argument, see Blakers and Massey, "Homotopy groups of Triads" I, II, III", Annals of Mathematics Vol. 53, 55, 58.

81 **4.5.4** If P is $(a - 1)$-connected, Q is $(b - 1)$-connected, then $P * Q$ is $(a + b)$-connected.

For, let $C(P)$, $C(Q)$ be cones with vertices v, w. Then $(C(P), -v)$ is a-connected, $(C(Q), -w)$ is b-connected. Hence by 4.5.3, $(C(P) \times C(Q), -(v, w))$ is $(a+b+1)$-connected. By 4.3.19, this pair is equivalent to $(C(P*Q), -(v, w))$. Hence $P*Q$ is $(a+b)$-connected. For a direct proof of 4.5.4, see Milnor's "Construction of Universal Bundles II (Annals of Mathematics, 1956, Vol. 63).

4.5.5 The join of K non-empty polyhedra is $(k - 2)$-connected. In particular $(k - 1)$-sphere is $(k - 2)$-connected. The join of a $(k - 1)$-sphere and a a-connected polyhedron is $(a + k)$-connected. Thus a k^{th} suspension (same as the join with a $(k - 1)$-sphere) of a connected polyhedron is at least K-connected.

Chapter 5

General Position

We intend to study PL-manifolds is some detail. There are certain basic techniques which have been developed for this purpose, one of which is called "general position". An example is the assertion that "if K is a complex of dimension k, M a PL-manifold of dimension $> 2k$, and $f : K \to M$ is any map, then f can be approximated by imbeddings". More generally we start with some notions "a map $f : K \to M$ being generic" and "a map $f : K \to M$ being in "generic position" with respect to some $Y \subset M$". This "generic" will be usually with reference to some minimum possible dimensionality of "intersections", "self intersections" and "nicety of intersections". The problem of general position is to define useful generic things, and then try to approximate nongeneric maps by generic ones for as large a class of X's, Y's and M's as possible (even in the case of PL-manifolds, one finds it necessary to prove general position theorems for arbitrary K). **82**

It seems that the first step in approximating a map by such nice maps is to approximate by *a* so called nondegenerate map, that is a map $f : K \to M$ which preserves dimensions of subpolyhedra.

Now it happens that a good deal of 'general position' can be obtained from just this nondegeneracy, that is if Y is the sort of polyhedron in which maps from polyhedra of dimension $\leq$ some n can be approximate by nondegenerate maps, then they can be approxiamted by nicer maps also. And the class of these Y's is much larger than that of PL-manifolds. **83**

We call such spaces Non Degenerate (n)-spaces or $ND(n)$-spaces. The aim of this chapter is to obtain a good description of such spaces and prove a few general position theorems for these spaces.

5.1 Nondegeneracy

Proposition 5.1.1. *The following conditions on a polyhedral map $f : P \to Q$ are equivalent:*

(a) *For every subpolyhedron X of P,*

$\dim f(X) = \dim X$.

(b) *For every subpolyhedron Y of Q,*

$\dim f^{-1}(Y) \leq \dim Y$.

(c) *For every point $x \in Q$, $f^{-1}(x)$ is finite.*

(d) *For every line segment $[x, y] \subset P$, $x \neq y$, $f([x, y])$ contains more than one point.*

(e) *For every $\mathscr{P}$, $\mathcal{Q}$ with respect to which f is simplicial, $f(\sigma)$ has the same dimension as σ, $\sigma \in \mathscr{P}$.*

(f) *There exists a presentation $\mathscr{P}$ of p, on each cell of which f is linear, and one-to-one.*

Proof. Clearly

$$
\begin{aligned}
&(a) \implies (d) \\
&(b) \implies (c) \implies (d) \\
&(e) \implies (f)
\end{aligned}
$$

84 To see that (a) $\implies$ (b):

Consider a subpolyhedron Y of Q; then $f(f^{-1}(Y)) \subset Y$. $\mathrm{Dim}(f^{-1}(Y)) = \dim f(f^{-1}(Y))$ by (a) and as $f(f^{-1}(Y)) \subset Y$, $\dim f(f^{-1} * Y)) \leq \dim Y$. Hence $\dim(f^{-1}(Y) \leq \dim Y$.

To see the (d) $\Rightarrow$ (e):

Let $\sigma \in \mathscr{P}$. If $f(\sigma)$ has not the same dimension as that of σ, two different vertices of σ say v_1 and v_2 are mapped onto the same vertex of $f(\sigma)$ say v. Then $[v_1, v_2]$ is mapped onto a single point v, contradicting (d).

Finally (f) $\Rightarrow$ (a):

To See this, first observe that if f is linear and one-to-one- on a cell C, then it is linear one one-to-one on $\overline{C}$ also. Thus if A is a polyhedron in $\overline{C}$, $\dim f(A) = \dim A$. But, $X = \bigcup_{C \in \mathscr{P}} (X \cap \overline{C})$, and $\dim X = \mathrm{Max}_{C \in \mathscr{P}} (\dim X \cap \overline{C})$. It follows that $\dim f(X) = \dim X$.

Thus we have

$$\begin{tikzcd}
& b \arrow[dr, Rightarrow] & \\
a \arrow[ur, Rightarrow] & & c \arrow[d, Rightarrow] \\
f \arrow[u, Rightarrow] & & d \arrow[dl, Rightarrow] \\
& e \arrow[ul, Rightarrow] &
\end{tikzcd}$$

and therefore all the conditions are equivalent. $\square$

Definition 5.1.2. We shall call a polyhedral map f which satisfies any of the six equivalent conditions of proposition 5.1.1. *a nondegenerate* map.

Note that a nondegenerate map may have various "foldings"; in **85**
other words it need not be a local embedding.

Ex. 5.1.3. (1) If $f : P \to Q$ is a polyhedral map, and $P = P_1 \cup \ldots \cup P_k$, P_i is a subpolyhedron of P, $1 \leq i \leq k$, and if f/P_i is nondegenerate, then f is nondegenerate.

(2) If $f : P \to Q$ is nondegenerate, and $X \subset P$ a subpolyhedron, then f/X is also nondegenerate.

[Hint : Use 1.C].

Ex. 5.1.4. Proposition. The composition of two nondegenerate maps is a nondegenerate map.

Ex. 5.1.5. Proposition. If $f : P_1 \to Q_1$, and $g : P_2 \to Q_2$ are nondegenerate, then $f * g : P_1 * P_2 \to Q_1 * Q_2$ is nondegenerate.

In Particular conical extensions of nondegenerate maps are again nondegenerate.

[Hint: Consider presentations with respect to which f, g are simplicial and use 1 f.].

Let $f : P \to Q$ be a polyhedral map, $\mathscr{S}$ and $\mathscr{K}$ triangulations of P and Q with reference to which f is simplicial. $\mathscr{S}_k$ and $\mathscr{K}_k$ as usual denote the k^{th} skeletons of $\mathscr{S}$ and $\mathscr{K}$. Let θ, η be centerings of $\mathscr{S}$, $\mathscr{K}$ respectively such that $f(\theta\sigma) = \eta(f\sigma)$ for $\sigma \in \mathscr{S}$. Let $\mathscr{S}^k$ and $\mathscr{K}^k$ denote the dual skeletons with respect to these centerings. Then

Ex. 5.1.6. (a) $f(\mathscr{S}_k) \subset \mathscr{K}_k$.

f is nondegenerate if and only if $f^{-1}(\mathscr{K}_k) \subset \mathscr{S}_k$.

86 (b) $f^{-1}(\mathscr{K}^k) \subset \mathscr{S}^k$

f is nondegenerate if and only if

$f(\mathscr{S}^k) \subset \mathscr{K}^k$.

(c) Formulate and prove the analogues of (a)

and (b) for regular presentations.

5.2 $ND(n)$-spaces. Definition and Elementary properties

Definition 5.2.1. A polyhedron M is said to be an $ND(n)$-*space (read Non-Degenerate* (n)-*sace)* if and only if:

for every polyhedron X of dimension $\leq n$, and any map $f : X \to M$ and any $\epsilon > 0$, there is an ϵ-approximation to f which is nondegenerate.

This property is a polyhedral invariant:

Proposition 5.2.2. *If M is and $ND(n)$-space, and $\mathcal{L} : M \to M'$ a polyhedral equivalence, then M' is also $ND(n)$.*

Proof. Obvious. □

Before we proceed further, it would be nice to know such spaces exist. Here is an example:

Proposition 5.2.3. *An n-cell is an $ND(n)$-space.*

Proof. By 5.2.2, it is enough to prove for $\overline{A}$, where A is an open convex n-cell in $\mathbb{R}^n$. Let $f : X \to \overline{A}$ be any map from a polyhedron X of dimension $\leq n$. First choose a triangulation $\mathscr{S}$ of X, such that f is linear on each simplex of $\mathscr{S}$. Let $v_1, \ldots, v_r$ be the vertices of $\mathscr{S}$. First we alter the map f a little to a f' so that $f'(v_1), \ldots, f'(v_r)$ are all in A = Interior of $\overline{A}$. This is clearly possible: We just have to choose points near **87**
$f(v_i)$'s in the interior and extend linearly. Next, by 1.2.12 of Chapter 1, we can choose $y_1, \ldots, y_r$ so that y_i is near $f'(v_i)$ and y_i's are in general position, that is any $(n+1)$ or less number of points of y's is independent. If we choose y's near enough $f'(v_i)$'s, the y's will be still in $\overline{A}$, that is why we shifted $f(v_i)$'s into the interior. Now we define $g(v_i) = y_i$ and extend linearly on simplexes of $\mathscr{S}$ to a get a map $X \to M$, which is non-degenerate by 5.1.1 (f). And surely if $f(v_i)$ and y_i are near enough, g will be good approximation to f. □

The next proposition says, roughly, that an $ND(n)$-space is locally $ND(n)$.

Proposition 5.2.4. *If $\mathscr{S}$ is any simplicial presentation of an $ND(n)$-space, and $\sigma \in \mathscr{S}$, then $|St(\sigma, \mathscr{S})|$ is an $ND(n)$-space.*

Proof. If x is a point of σ, then $|St(\sigma, \mathscr{S})|$ is a cone with vertex x and base $\partial\sigma * |Lk(\sigma, \mathscr{S})|$ which is a link of x in M; and $|St(\sigma, \mathscr{S})| - \partial\sigma * |Lk(\sigma, \mathscr{S})|$ is open in M. If $f : X \to |St(\sigma, \mathscr{S})|$ is any map from a polyhedron X of dimension $\leq n$ and $\epsilon > 0$, we first shink it towards x by a map f' say so that $f'(X) \subset |St(\sigma, \mathscr{S})| - \partial\sigma * |Lk(\sigma, \mathscr{S})|$ so that $\rho(f, f') < \epsilon/2$. Now $N = M - (|St(\sigma, \mathscr{S})\min - \partial\sigma * |Lk(\sigma, \mathscr{S})|)$ is a sub-polyhedron of M, and $f'(X) \cap N = \emptyset$. Therefore $\rho(f'(X), N) > \delta > 0$. Let $\eta = \min(\delta, \epsilon/2)$. Since M is $ND(n)$, we can obtain an η-approximation to f', say g which is nondegenerate. g is an ϵ-approximation to f and $g(X) \cap N = \emptyset$, $g(X) \subset M$. Therefore $g(X) \subset | St(\sigma, \mathscr{S})|$. Hence **88**
$|St(\sigma, \mathscr{S})|$ is an $ND(n)$-space. □

Next we establish a sort of "general position" theorem for $ND(n)$-space.

Theorem 5.2.5. *Let M be an $ND(n)$-space, K a subpolyhedron of M of dimension $\leq k$. Let $f : X \to M$ be a map from a polyhedron X of dimension $\leq n-k-1$. Then f can be approximated by a map $g : X \to M$ such that $g(X) \cap K = \emptyset$.*

Proof. Let D be a $(k+1)$-cell. Let $f' : D \times X \to M$ be the composition of the projection $D \times X \to X$ and f, that is $f'(a,x) = f(x)$ for $a \in D$, $x \in X$. By hypothesis, $\dim(D \times X) \leq n$. Hence f' can be approximated by a map g' which is nondegenerate. The dimension of $g'^{-1}(K) \leq k$. Consider $\pi(g'^{-1}(K))$; (where π is the projection $D \times X \to D$), this has dimension $\leq k$; hence it cannot be all of the $(k+1)$-dimensional D. Choose some $a \in D - \pi(g'^{-1}(K))$. Then $g'(a \times X) \cap K = \emptyset$. We define g by, $g(x) = g'(a,x)$, for $x \in X$. Since $f(x) = f'(a,x)$, and g' can be chosen to be as close to f' as we like, we can get a g as close to f as we like. □

We can draw a few corollaries, by applying the earlier approximation theorems.

Ex. 5.2.6. If M is $ND(n)$, K a subpolyhedron of M of dimension $\leq k$, then the pair $(M, M-K)$ is $(n-k-1)$-connected.

[Hint: It is enough to consider maps $f : (D, \partial D) \to (M, M-K)$, and show that such an f is homotopic to a map g by a homotopy which is
89 fixed on ∂D, and with $g(D) \subset M - K$. First, by 5.2.5, one can get a very close approximation g_1 to f with $g_1(D) \subset M - K$. Then since $g_1|\partial D$ and $f|\partial D$ are very close, there will be a small homotopy h (3.2.3) in a compact polyhedron in $M-K$ with $h_0 = f|\partial D$, $h_1 = g_1|\partial D$. Expressing D as the identification space of $\partial D \times I$ and D_1 (a cell with $\partial D_1 = \partial D \times 1$) at $\partial D \times 1$ and patching up h and the equivalent of g_1 on D_1, we get a map $g : D \to M$, with $g|\partial D = f|\partial D$, $g(D) \subset M-K$ and g close to f. Then there will a homotopy of f and g fixed on ∂D].

As an application this and 5.2.4 we have:

Proposition 5.2.7. *If $\mathscr{S}$ is a simplicial presentation of an $ND(n)$-space and $\sigma \in \mathscr{S}$, then $|Lk(\sigma, \mathscr{S})|$ is $(n - \dim \sigma - 2)$-connected.*

Proof. For by 5.2.4, $|St(\sigma, \mathscr{S})| = \overline{\sigma} * |Lk(\sigma, \mathscr{S})|$ is $ND(n)$, and by 5.2.6, $(|St(\sigma, \mathscr{S})|, |St(\sigma, \mathscr{S})| - \overline{\sigma})$ is $(n - \dim \sigma - 1)$-connected, thus giving that $|St(\sigma, \mathscr{S})| - \overline{\sigma}$ is $(n - \dim \sigma - 2)$-connected. But $|Lk(\sigma, \mathscr{S})|$ is a deformation retract of $|St(\sigma, \mathscr{S})| - \overline{\sigma}$. □

5.3 Characterisations of $ND(n)$-spaces

We now introduce two more properties: the first an inductively defined local property called $A(n)$ and the second a property of simplicial presentations called $B(n)$ and which is satisfied by the simplicial presentations of $ND(n)$-spaces. It turns out that if M is a polyhedron and $\mathscr{S}$ a simplicial presentation of M, then M is $A(n)$ if and only if $\mathscr{S}$ is $B(n)$. Finally, we complete the circle by showing that $A(n)$-space have an approximation property which is somewhat stronger than that assumed for **90**
$ND(n)$-spaces. $A(n)$ shows that $ND(n)$ is a local property. $B(n)$ is useful in checking whether a given polyhedron is $ND(n)$ or not. Using these, some more descriptions and properties of $ND(n)$-spaces can be given.

Definition 5.3.1 (The property $A(n)$ for polyhedra)**.**
Any polyhedron is $A(0)$.
If $n \geq 1$, a polyhedron M is $A(n)$ if and only if the link of every point in M is a $(n-2)$-connected $A(n-1)$.

Definition 5.3.2 (The property $B(n)$ for simplicial presentations)**.**
A simplicial presentation $\mathscr{S}$ is $B(n)$, if and only if for every $\sigma \in \mathscr{S}$, $|Lk(\sigma, \mathscr{S})|$ is $(n - \dim \sigma - 2)$-connected.

By 5.2.7, we have

Proposition 5.3.3. *If M is $ND(n)$, then every simplicial presentation $\mathscr{S}$ of M is $B(n)$.*

The next to propositions show that $A(n)$ and $B(n)$ are equivalent (ignoring logical difficulties).

Proposition 5.3.4. *If M is $A(n)$, then every simplicial presentation of M is $B(n)$.*

Proof. The proof is by induction on n. For $n = 0$, the $B(n)$ condition says that certain sets are (≤ -2)-connected, i.e. any $\mathscr{S}$ is $B(0)$, agreeing with the fact that any M is $A(0)$. Let $n > 0$, and assume the proposition for $m < n$.

Let $|\mathscr{S}| = M$, $\sigma \in \mathscr{S}$ and $\dim \sigma = k$.

If $k = 0$, then by the condition $A(n)$, the link of the element of σ,
91 which can be taken to be $|Lk(\sigma, \mathscr{S})|$ is $(n-2)$-connected.

If $k > 0$, let x be any point of σ. Then a link of x in M is $\partial\sigma * |Lk(\sigma, \mathscr{S})|$, which is $A(n-1)$ by hypothesis.

Hence by inductive hypothesis, its simplicial presentation $\{\partial\sigma\} * Lk(\sigma, \mathscr{S})$ satisfies $B(n-1)$. If τ is any $(k-1)$-dimensional face of σ,

$$|Lk(\sigma, \mathscr{S})| = |Lk(\tau, \{\partial\sigma\} * Lk(\sigma, \mathscr{S}))|$$

which is $((n-1)-(k-1)-2)$-connected i.e. $(n-k-2)$-connected since $\{\partial\sigma\} * Lk(\sigma, \mathscr{S})$ is $B(n-1)$. □

Proposition 5.3.5. *If a polyhedron M has a simplicial presentation $\mathscr{S}$ which is $B(n)$, then M is $A(n)$.*

Proof. The proof is again by induction. For $n = 0$, it is the same as in the previous case. And assume the proposition to be true for all $m < n > 0$.

Let $x \in M$. Then x belongs to some simplex σ of $\mathscr{S}$, and a link of x in M is $\partial\sigma * |Lk(\sigma, \mathscr{S})|$. We must show that this is an $(n-2)$-connected $A(n-1)$.

As per connectivity, we note (setting $k = \dim \sigma$) that $\partial\sigma$ is a $(k-1)$-sphere; and by $B(n)$, $|Lk(\sigma, \mathscr{S})|$ is $(n-k-2)$-connected. As the join with a $(k-1)$-sphere rises connectivity by k, $\partial\sigma * |Lk(\sigma, \mathscr{S})|$ is $(n-2)$-connected.

To prove that $\partial\sigma * |Lk(\sigma, \mathscr{S})|$ is A_{n-1}, it is enough to show that $\{\partial\sigma\} * Lk(\sigma, \mathscr{S}) = \mathscr{S}'$ say is $B(n-1)$; for then by induction it would follow that $|\mathscr{S}'| = \partial\sigma * |Lk(\sigma, \mathscr{S})|$ is $A(n-1)$. Take a typical simplex
92 $\mathcal{L}$ of $\mathscr{S}'$. It is of the form $\beta\gamma$, $\beta \in \{\partial\sigma\}$, $\gamma \in Lk(\sigma, \mathscr{S})$, with β or $\gamma = \emptyset$ being possible. Now $Lk(\mathcal{L}, \mathscr{S}') = Lk(\beta, \{\partial\sigma\}) * Lk(\gamma, Lk(\sigma, \mathscr{S}))$.

Let a, b, c be the dimensions of α, β, γ respectively. $a = b + c + 1$. Remember that $\dim \sigma = k$. Therefore $|Lk(\beta, \{\partial\sigma\})|$ is a $(k-b-2)$-sphere. Now $|Lk(\gamma, Lk(\sigma, \mathscr{S}))| = |Lk(\gamma\sigma, \mathscr{S})|$; and by $B(n)$ assumption, this is $(n-(c+k+1)-2)$ connected. Hence the join of $|Lk(\beta, \{\partial\sigma\})|$ and $|Lk(\gamma, Lk(\sigma, \mathscr{S}))|$ which is $|Lk(\alpha, \mathscr{S}')|$ is

$$[(n-(c+k+1)-2)+(k-b-2)+1]\text{-connected}$$

that is $((n-1)-a-2)$-connected.

Thus $\mathscr{S}'$ is $B(n-1)$, and therefore by induction $|\mathscr{S}'| = \partial\sigma * |Lk(\sigma, \mathscr{S})|$, a link of x in M is a $(n-2)$-connected $A(n-1)$. Hence M is $A(n)$. □

We need the following proposition for the next theorem.

Proposition 5.3.6. *Let $\mathscr{P}$ be a regular presentation of an $A(n)$-space M and η be any centering of $\mathscr{P}$. Let A be any element of $\mathscr{P}$, and* $\dim A = k$. *Then*

$$|\lambda A| \textit{ is an } (n-k-2)\textit{-connected} \quad A_{n-k-1}$$

and $|\delta A|$ is a contractible A_{n-k}.

Proof. We know that λA is the link of a k-simplex in $d\mathscr{P}$. Since $d\mathscr{P}$ satisfies $B(n)$, $|\lambda A|$ is $(n-k-2)$-connected.

If $k = 0$, λA is the link of a point and therefore $A(n-1)$, since M is $A(n)$.

If $k > 0$, then $\partial A * |\lambda A|$ is a link of a point in M, and so is $A(n-1)$. 93 Take a $(k-1)$-simplex σ of $d\mathscr{P}$ in ∂A; then λA is $Lk(\sigma, d\{\partial A\} * \lambda A)$ which (by induction on k), we know to be a presentation of an $A((n-1)-(k-1)-1)$-space.

To prove that $|\delta A|$ is $A(n-k)$, we prove that δA is $B(n-k)$. Consider its vertex ηA, then $Lk(\eta A, \delta A) = \lambda A$, and $|\lambda A|$ is $(n-k-2)$-connected. For a simplex $\sigma \in \lambda A$, we have $|Lk(\sigma, \delta A)| =\subset |(Lk(\sigma, \lambda A))|$ which is contractible. For a simplex $\tau = \sigma\{\eta A\}$, $\sigma \in \lambda A$, $Lk(\tau, \delta A) = Lk(\sigma, \lambda A)$. If τ has dimension t, σ has dimension $(t-1)$; and so $|\lambda A|$ being $A(n-k-1)$, $|Lk(\sigma, \lambda A)|$ is $((n-k-1)-(t-1)-2)$-connected, i.e. $|Lk(\tau, \delta A)|$ is $((n-k)-t-2)$-connected. This shows that δA satisfies $B(n-k)$. □

Theorem 5.3.7. *Let M be an $A(n)$-space, $Y \subset X$ polyhedra of dimension $\leq n$, and $f : X \to M$ a map such that $f|Y$ is non-degenerate. Given any $\epsilon > 0$, there is an ϵ-approximation g to f such that g is nondegenerate and $g|Y = f|Y$.*

Proof. The proof will be by induction on n. If $n = 0$, we take $g = f$, since any map on a 0-dimensional polyhedron is nondegenerate.

So assume $n > 0$, that the proposition with m instead of n to be true for all $m < n$.

Without loss of generality we can assume that f is polyhedral. Choose simplicial presentations $\mathscr{Z} \subset \mathscr{S}$, $\mathscr{M}$ of Y, X and M such that f is simplicial with respect to $\mathscr{S}$ and $\mathscr{M}$; and such that the diameter of the star of each simplex in $\mathscr{M}$ is less than ϵ. Let θ, η be center-
94 ings of $\mathscr{S}$ and $\mathscr{M}$ with $f(\theta\sigma) = \eta(f\sigma)$ for all $\sigma \in \mathscr{S}$. Then clearly $f^{-1}(\mathscr{M}^k) \subset \mathscr{S}^k f(\mathscr{Z}^k) \subset \mathscr{M}^k$ and the diameter of $|\delta\rho|$ is less than ϵ for every $\rho \in \mathscr{M}$.

Consider an arrangement $A_1, \ldots, A_r$ of simplexes of $\mathscr{M}$ so that $\dim A_i \geq \dim A_{i+1}$, for $1 \leq i \leq k$. The crucial fact about such an arrangement is, for each i, (*) λA_i is the union of δA_j for some j's less than i.

We construct an inductive situation $\sum_i$ such that

(1) $X_i = f^{-1}(|\delta A_1| \cup \ldots \cup |\delta A_i|)$

(2) $Y_i = X_i \cap Y$

(3) $g_i : X_i \to M$, a nondegenerate map

(4) $g_i(f^{-1}|\delta A_i|) \subset |\delta A_i|$

(5) $g_i|X_{i-1} = g_{i-1}$

(6) $g_i|Y_i = f|Y_i$

$\mathscr{M}^n$ is the union of certain A_i's in the beginning, say A_i's with $i \leq \ell$. $f^{-1}(\mathscr{M}^n) \subset \mathscr{S}^n$ and $|\mathscr{S}^n|$ is 0-dimensional. Hence $f|f^{-1}(|\delta A_1| \ldots |\delta A_\ell|)$ is already nondegenerate. If we take this to be g_ℓ all the above properties are satisfied an we have more than started the induction. Now let $i > \ell$

and suppose that g_{i-1} is defined, that is we already have the situation $\sum_{(i-1)}$.

It follows from (4) and (5), that for $j < i$, $g_{i-1}(f^{-1}|\delta A_j|) \subset |\delta A_j|$, and hence from (*) that g_{i-1} maps $f^{-1}(|\lambda A_i|)$ into $|\lambda A_i|$.

Also this shows that if $x \in f^{-1}(|\delta A_j|)$ then both $g_{i-1}(x)$ and $f(x)$ are 95
in $|\delta A_j|$, which has diameter $< \epsilon$, and so g_{i-1} is an ϵ-approximation to $f|X_{i-1}$.

There are ow two cases.

Case 1. $\dim A_i = k \geq 1$.

Look at $|\delta A_i|$. This is a contractible $A(n-k)$. Let $X' = f^{-1}(|\delta A_i|)$ and $Y' = (Y \cap X') \cup f^{-1}(|\lambda A_i|)$. The maps f on $Y \cap X'$ and g_{i-1} on $f^{-1}(|\lambda A_i|)$ agree where both are defined by $\sum_{i-1}(6)$, and are nondegenerate by hypothesis and induction. Hence patching them up we get a nondegenerate map $f' : Y' \to |\delta A_i|$. Since $|\delta A_i|$ is contractible f' can be extended to a map (still denoted by f') of X' to $|\delta A_i|$. Since $X' \subset |\mathscr{S}^k|$, $\dim X' \leq n-k$, and $|\delta A_i|$ is $A(n-k)$, there is a nondegenerate map $f'' : X' \to |\delta A_i|$ such that $f''|Y' = f'|Y'$, by using the theorem for $(n-k) \leq n-1$.

We now define g_i to be g_{i-1} on X_{i-1} and f'' on X'; these two maps agree where both are defined, namely $f^{-1}(|\lambda A_i|)$. Thus g_i is well defined and is nondegenerate as both f'' and g_{i-1} are nondegenerate. And clearly all the six conditions of $\sum_i$ are satisfied.

Case 2. $\operatorname{Dim} A_i = 0$.

Let $B_1, \ldots, B_s$ be the vertices of $\mathscr{S}$ which are mapped onto A_i. Then

$$f^{-1}(|\delta A_i|) = |\delta B_1| \cup \ldots |\delta B_s|.$$

$$\text{Let} \qquad X' = |\lambda B_1| \cup \ldots \cup |\lambda B_s| = |\mathscr{S}^1| \cap f^{-1}(|\delta A_i|)$$

X' is of dimension $\leq n-1$, and contains $f^{-1}(|\lambda A_i|)$. Let $Y' = f^{-1}(|\lambda A_i|)$. 96
Here the important point to notice is, that $Y \cap X' \subset Y'$. This is because $f|Y$ is nondegenerate: $Y \cap X' \subset \mathscr{X}^1$, so $f(Y \cap X') \subset \mathscr{M}^1$. It is also in $|\delta A_i|$ and therefore $f(Y \cap X') \subset |\mathscr{M}^1| \cap |\delta A_i| = |\lambda A_i|$.

We first extend $g_{i-1}|Y'$ to X' and then by conical extension to f^{-1} $(|\delta A_i|)$. g_{i-1} maps Y' into $|\lambda A_i|$ and is non-degenerate on Y'. Since $|\lambda A_i|$

is $(n-2)$-connected, and $\dim X' \leq n-1$, $g_{i-1}|Y'$ can be extended to a map f' of X' into $|\lambda A_i|$ $|\lambda A_i|$ is also $A(n-1)$. Hence by the inductive hypothesis we can approximate f' by a nondegenerate map f'' such that $f''|Y' = f'|Y' = g_{i-1}|Y'$.

Hence $f''||\lambda B_j|, 1 \leq j \leq s$ is nondegenerate and maps $|\lambda B_j|$ into $|\lambda A_i|$. We extend this to a map $h_j : |\delta B_j| \to |\delta A_i|$, by mapping B_j to A_i and taking the join is clearly nondegenerate. Since $|\delta B_j| \cap |\delta B_{j'}| \subset |\lambda B_j| \cap |\lambda B_{j'}| \subset X'$, if $j \neq j'$, h_j's agree whereever their domains of definition overlap. Similarly h_j and g_{i-1} agree where both are defined. We now define g_i to be g_{i-1} on X_{i-1} and h_j on $|\delta B_j|$. Thus g_i is defined on $X_{i-1} \cup f^{-1}(|\delta A_i|) = X_i$ and is nondegenerate since g_{i-1} and h_j's are nondegenerate. It obviously satisfies conditions 1-5 of $\sum_i$, to see that it satisfies (6) also: Let σ is any simplex of $d\mathscr{X}$ in δB_j, if B_j is not a vertex of σ there is nothing to prove; if B_j is a vertex of σ, write $\sigma = \{B_j\}\sigma'$. Both h_j and f agree on $\overline{\sigma}'$ and B_j and on $\overline{\sigma}$ both are joins, hence both are
97 equal on $\overline{\sigma}$. Then (6) is also satisfied and we have the situation $\sum_i$. $\square$

This theorem shows in particular that $ND(n)$ is a local property; and that $ND(n)$-spaces have stronger approximation property than is assumed for them.

The following propostions, which depend on the computations of links are left as exercises.

Ex. 5.3.8. Proposition. $C(X)$ and $S(X)$ are $ND(n)$ if and only if X is an $(n-2)$-connected $ND(n-1)$.

Thus the k^{th} suspension of X is $ND(n)$ if and only if X is an $(n-k-1)$-connected $ND(n-k)$-space.

Ex. 5.3.9. Proposition. Let $\mathscr{S}$ be a simplicial presentation of X. Then X is $ND(n)$ if and only $|Lk(v, \mathscr{S})|$ is $(n-2)$-connected $ND(n-1)$ for each vertex v of $\mathscr{S}$.

Ex. 5.3.10. Proposition. If $\mathscr{S}$ is a simplicial presentation of an $ND(n)$-space, and $0 \leq k \leq n$, then the skeleton $\mathscr{S}_k$ is $ND(k)$, and the dual skeleton $\mathscr{S}^k$ is $ND(n-k)$.

Thus the class of $ND(n)$-spaces is much larger than the class of PL n-manifolds, which incidentally are $ND(n)$ by the $B(n)$-property.

The results of this section can be summarised in the following proposition:

Proposition 5.3.11. *The following conditions on a polyhedron M are equivalent:*

(1) *M is $ND(n)$*

(2) *M is $A(n)$*

(3) *a simplicial presentation of M is $B(n)$* **98**

(4) *every simplicial presentation of M is $B(n)$*

(5) *there exists a simplicial presentation $\mathscr{S}$ of M such that $|LK(v, \mathscr{S})|$ is $(n-2)$-connected A_{n-1} for all $v \in \mathscr{S}$ and* $\dim v = 0$

(6) *M satisfies the approximation property of theorem 5.3.7.*

(7) *$M \times I$ is $ND(n+1)$.*

5.4 Singularity Dimension

5.4.1 Definitions and Remarks Let P and M be two polyhedra, $\dim P = p$, $\dim M = m$, p m, and $f : P \to M$ a nondegenerate map. Ed define the singularity of f (or the 2-fold singularity of f) to be set $\{x \in P | f^{-1} f(x)$ contains at least 2 points$\}$, and denote it by $S(f)$ or $S_2(f)$. By triangulating f, it can be seen easily that $S(f)$ is a finite union of open cells, so that $\overline{S(f)}$ is a subpolyhedron of p.

Similarly, we define the *r-fold singularity of f* for $r \geq 3$, to be the set $\{x \in P | f^{-1} f(x)$ contains at least r points$\}$. This will be denoted by $S_r(f)$. As above $S_r(f)$ is a finite union of open cells, so that $\overline{S_r(f)}$ is a subpolyhedron of P. Clearly $S2(f) \supset S_3(f)) \ldots$; and $S_r(f)$ are empty after a certain stage; since f is nondegenerate.

The number $(m - p)$ is usually referred to as the *codimension*; and the number $r(p) - (r - 1)m$, for $r \geq 2$ is called the *r-fold point dimension* and is denoted by d_r (see e.g. Zeeman "Seminar on combinatorial Topology", Chapter VI). Clearly $d_r = d_{r-1} - (m - p)$.

99 It will be convenient to use the notions of dimension and imbedding in the following cases: (1) dimension of A, where A is a union of open cells. In this case the dim. A denotes the maximum of the dimensions of the open cells comprising A and is the same as the dimension of the polyhedron $\overline{A}$. (2) Imbedding f of $C \to M$, when C is an open cell and M a polyhedron. This will be used only when f comes from a polyhedral embedding of $\overline{C}$. In such a case $f(C)$ will be the union of a finite member of open cells. And if $A \subset M$ is some finite union of open cells, then $f^{-1}(A)$ will be finite union of open cells and one can talk of its dimension etc..

A nondegenerate map $f : P \to M$ will be said to be in *general position* if

$$\dim(S_r(f)) \leq d_r, \quad \text{for all} \quad r$$

If $p = m$, this means nothing more than that f is nondegenerate, so usually $p < m$.

Proposition 5.4.2. *Let $\mathscr{P}$ be a regular presentation of a polyhedron P such that for every $C \in \mathscr{P}$, $f|C$ is an embedding. Let the cells of $\mathscr{P}$ be $C_1, \ldots, C_t$, arranged so that* $\dim C_i \leq \dim C_{i+1}$, $1 \leq i \leq t$, *and let P_i, $i \leq t$ be the subpolyhedron of P whose presentation is* $\{C_1, \ldots, C_i\}$. *Then*

(i) $$S_2(f|P_i) = S_2(f|P_{i-1}) \cup \{C_i \cap f^{-1}(f(P_{i-1}))\} \cup \{P_{i-1} \cap f^{-1}(f(C_i))\}$$

(ii) $$S_r(f|P_i) = S_r(f|P_{i-1}) \cup \{C_i \cap f^{-1}(f(S_{r-1}(f|P_{i-1}))\} \cup \{S_{r-1}(f|P_{i-1}) \cap f^{-1}(f(C_i))\}$$

100 This is obvious. If we write $P = S_1(f)$, (compatible with the definition of S_r's, then $S_1(f\ P_i)$ would be just P_i, and only (ii) be written (with $r \geq 2$) instead of (i) and (ii).

The proposition is useful in inductive proofs. For example, to check that a nondegenerate f is in general position, it is enough check for each little cell C_i, that $\dim C_i \cap f^{-1}(f(S_{r-1}(f|P_{i-1})) \leq d_r$. If we have already checked upto the previous stage; since f is non-degenerate $f^{-1}f(S_{r-1}(f|P_{i-1}))$ will of dimension $d_{(r-1)}$, and then we will have to verify that $f\ C_i$ intersects $f(S_{r-1}(f|P_i)))$ in codimension $\geq (m-p)$ or that (C_i) intersects $f^{-1}f(S_{r-1}(f|P_i))$ in codimension $\geq m-p$, (We usually say that *A intersects B in codimension q* if $\dim(A \cap B) = \dim B - q$. Similarly the expression '*A intersects B in codimension* $\geq q$' is used to denote $\dim(A \cap B) \leq \dim B - q$). The aim of the next few propositions is to obtain presentations on which it would be possible to inductively change the map, so that $f(C_i)$ will intersect the images of the previous singularities in codimension $\geq (m-p)$. Proposition 5.4.7 and 5.4.9 are ones we need; the others are auxilary to these.

Ex. 5.4.3. Let A, B, C, be three open convex cells, such that $A \cap B$ is a single point and $C \supset A \cup B$. Then $\dim C \geq \dim A + \dim B$.

[**Hint:** First observe that if A' and B' are any two *intersecting* open cells then $L_{A'} \cap L_{B'} = L_{A' \cap B'}$, where L_X denotes the linear manifold spanned by X. Applying this to the above situation

$$\begin{aligned}\dim C = \dim L_C \geq \dim L_{(A\cup B)} &= \dim L_A + \dim L_B - \dim(L_A \cap L_B)\\ &= \dim L_A + \dim L_B - \dim(L_{A\cap B})\\ &= \dim L_A + \dim L_B,\ \text{since } A \cap B\end{aligned}$$

is a point.] 101

Proposition 5.4.4. *Let A be an open convex cell of dimension n, and $\mathfrak{a}$ a regular presentation of $\overline{A}$ with $A \in \mathfrak{a}$. If L is any linear manifold such that* $\dim L \cap A = k \geq \theta$, *then there is a $B \in \mathfrak{a}$, of dimension $\leq n-k$, with $B \cap L \neq \emptyset$. Further, if A' is any cell of $\mathfrak{a}$ contained in ∂A, we can require that $\overline{A'} \cap B = \emptyset$.*

Proof. If $k = 0$, we can choose A itself to be B. If $k > 0$, consider the regular presentation $\mathscr{C} = \{C \cap L | C \cap L \neq \emptyset, C \in \mathfrak{a}\}$ of $\overline{A} \cap L$. $\mathscr{C}$ must have more than one 0-cell. Choose one of these 0-cells of C.

It must be the form $B \cap L$ for some $B \in \mathfrak{a}$. We would like to apply 5.4.2, for B, $L \cap A$ and A. But B and $L \cap A$ do not intersect. Since we are interested in the dimension of B, the situation can be remedied as follows: Let D be an n-cell, such that $\overline{A} \subset D$. $L \cap D$ is again k-dimensional. Since $B \subset D$, $B \cap L \subset D \cap L$, and as $B \cap L$ is nonempty, B and $D \cap L$ intersect. $B \cap (D \cap L)$ cannot be more than one point since $B \cap (D \cap L) \subset B \cap L$ which is just a point. Applying 5.4.2 to B, $D \cap L$ and D we have $n = \dim D \geq \dim B + \dim(D \cap L) = \dim B + k$, or, $\dim B \leq n - k$.

To see the additional remark, observe that all the vertices of $\mathscr{C}$ cannot be in $\overline{A'}$, for then $L \cap \overline{A} \subset \overline{A'}$, contrary to the hypothesis that $L \cap A$ is nonempty. Hence we can choose a 0-cell $B \cap L$, $B \in \mathfrak{a}$ of $\mathscr{C}$ not in $\overline{A'}$. Since $\mathfrak{a}$ is a regular presentation $B \cap \overline{A'} = \emptyset$. □

102 This just means that if L does not intersect the cells of $\mathfrak{a}$ of dim $\leq \ell$, then dimension of the intersection is $< n - \ell$, or codimension of intersectoin is $> \ell$. Using the second remark of 5.4.4 we have:

Corollary 5.4.5. *Let $\mathscr{P}$ be a regular presentation, containing a full subpresentation Q (which may be empty). Let $\mathscr{P}_k = \{\mathscr{C} \in \mathscr{P} - Q, \dim C \leq k\}$. If L is any linear manifold which does not intersect $\mathscr{P}_k$, then $\dim(L \cap (\mathscr{P} - Q) \leq n - k - 1$, where $n = \dim(\mathscr{P} - Q)$.*

Proposition 5.4.6. *Let A be a closed convex cell of dimension $\geq k + q$, let S be a $(k-1)$-sphere in ∂A: and $B_1, \ldots, B_r$ be a finite number of open convex cells of dimension $\leq q-1$ contained in the interior of A. Further, let $\mathscr{S}$ be a simplicial presentation of S. Then there is an open dense set U of interior A such that if $a \in U$, $\sigma \in \mathscr{S}$, then the linear manifold $L_{(\sigma,a)}$ generated by σ and 'a' does not intersect any of the B_i's.*

Proof. For any $\sigma \in \mathscr{S}$, consider the linear manifolds $L_{(\sigma,B_i)}$ generated by σ and B_i, for $1 \leq i \leq r$. $\operatorname{Dim} L_{(\sigma,B_i)} \leq k + q - 1$. Hence $U_\sigma = \operatorname{int} A - \bigcup\limits_i L_{(\sigma,B_i)}$ is an open dense subset of int A. If a is any point of U_σ, then $L_{(\sigma,a)}$ does not intersect any of the B_i's; for if there were a B_j with $L_{(\sigma,a)} \cap B_j \neq \emptyset$, let $b \in L_{(\sigma,a)} \cap B_j$. $L_{(\sigma,b)} \subset L_{(\sigma,a)}$ and is of the same dimension as $L_{(\sigma,a)}$, since b is in the interior of A. Thus

$a \in L_{(\sigma,b)} \subset L_{(\sigma,B_j)}$ contrary to the choice of a. Therefore if we take $U = \bigcap\limits_{\sigma\in\mathscr{S}} \bigcup_\sigma$, U satisfies our requirements. □

Proposition 5.4.7. *Let A be a closed convex cell of dimension $\geq k+q$, let S be a $(k-1)$-sphere contained in ∂A, and let $\{B_1,\dots,B_r\}$ be a finite number of open convex cells in int A. Then there is an open dense subset U of int A such that if $a \in U$, then $S * a$ intersects each of the B_i's in codimension $\geq q$.* **103**

Proof. Let $\mathscr{S}$ be some simplicial presentation of S. First let us consider one B_i. Let $\mathscr{B}_i$ be a regular presentation of $\overline{B}_i$ containing a full subpresentation $\mathscr{X}_i$ covering $\overline{B}_i \cap \partial A$. Let $\mathscr{B}_{q-1} = \{C \in \mathscr{B}_i - \mathscr{X}_i, \dim C \leq q-1\}$. By 5.4.6, there is an open dense subset of int A say U_i such that if $a \in U_i$, $\sigma \in \mathscr{S}$, then $L_{(\sigma,a)}$ does not intersect any of the elements of $\mathscr{B}_{q-1}$. By 5.4.5, $\dim L_{(\sigma,a)} \cap (\mathscr{B}_i - \mathscr{X}_i) \leq n_i - q$, where $n_i = \dim B_i$. Hence $\dim(S * a \cap B_i) \leq n_i - q$. Therefore if we take $U = \bigcap_j U_j$, where U_j constructed as above for each of B_j's, then U, satisfies the requirements of the proposition. □

Proposition 5.4.8. *Let σ be a k-simplex, Δ a closed convex q-cell; $\mathscr{P}$ a regular presentation of $\overline{\sigma} * \Delta$. Then there exists an open dense subset U of Δ, such that if $a \in U$, the linear manifold $L_{(\sigma,a)}$ spanned by σ and a, does not intersect any cell $C \in \mathscr{P}$ satisfying $C \cap \overline{\sigma} = \emptyset$ and* $\dim C \leq q-1$.

Proof. Let $C \in \mathscr{P}$, with $C \cap \overline{\sigma} = \emptyset$ and $\dim C \leq q-1$. The linear manifold $L_{(\sigma,C)}$ has dimension $\leq k+q$, while $L_{(\sigma,\Delta)}$ has dimension $k+q+1$. Therefore $L_{(\sigma,C)} \cap \Delta$ has dimension $\leq q-1$ and so $U_C = \Delta - L_{(\sigma,C)}$ is open and dense in Δ. Define U to be the intersection of all the U_C. If **104** $a \in U$, and there were some C of $\mathscr{P}$ of dimension $\leq q-1$, $C \cap \overline{\sigma} = \emptyset$, with $L_{(\sigma,a)} \cap C \neq \emptyset$, choose $b \in C \cap L_{(\sigma,a)}$; since $b \notin \overline{\sigma}$, $\dim L_{(\sigma,b)} = k+1 = \dim L_{(\sigma,a)}$ and so $L_{(\sigma,a)} = L_{(\sigma,b)}$ i.e. $L_{(\sigma,a)} \subset L_{(\sigma,C)}$, or, $a \in L_{(\sigma,C)}$ contrary to the choice of a. □

Proposition 5.4.9. *Let S be a $(p-1)$-sphere, Δ a closed convex q-cell, $\mathscr{P}$ a regular presentation of $S * \Delta$. Then there exists*

(1) *a regular refinement $\mathscr{P}'$ of $\mathscr{P}$*

(2) *a point $a \in \Delta$*

(3) *a regular presentation Q of $S * a$ such that*

  (a) *Q contains a full subpresentation $\mathscr{S}$ covering S,*

  (b) *Each $C \in Q - \mathscr{S}$ is the intersection of a linear manifold with a (unique) cell $E_C \in \mathscr{P}'$, if $C \neq C'$, $E_C \neq E'_C$, and if $C < C'$, then $E_C < E'_C$*

  (c) *$\dim C \leq \dim E_C - q$, for all $C \in Q - \mathscr{S}$.*

Proof. Let $\mathfrak{a}$, $\mathscr{B}$ be simplicial presentations of S, Δ; and let $\mathscr{P}'$ be a common simplicial refinement of $\mathfrak{a} * \mathscr{B}$ and $\mathscr{P}$. Since $\mathfrak{a}$ is full in $\mathfrak{a} * \mathscr{B}$, there is a subpresentation, say $\mathscr{S}$, of $\mathscr{P}'$ covering S. If $\sigma \in \mathfrak{a}$, $\overline{\sigma} * \Delta$ is covered by a subpresentation in $\mathfrak{a} * \mathscr{B}$, hence there is a subpresentation of $\mathscr{P}'$, say $\mathscr{P}'_\sigma$, covering $\overline{\sigma} * \Delta$. Applying 5.4.8 to $\mathscr{P}'_\sigma$, we get an open dense subset U_σ of Δ. Let U be the intersection of the sets U_σ for $\sigma \in \mathfrak{a}$.
105 Let $a \in U$. Obviously 'a' is in an (open) q-simplex of $\mathscr{P}'$ contained in Δ. Hence 'a' belongs to a q-simplex of $\mathscr{B}$, call it ρ.

We define Q to be union of $\mathscr{S}$, $\{a\}$, and all nonempty intersections of the form $L_{(\sigma,a)} \cap E$, for $\sigma \in \mathfrak{a}$, $E \in \mathscr{P}' - \mathscr{S}$. It is clear that $L_{(\sigma,a)} \cap E = \sigma\{a\} \cap E$. Moreover $\overline{E} \cap S = \overline{F}$, $F \in \mathscr{S}$ (F may be empty) since $\mathscr{S}$ is full in $\mathscr{P}'$. This immediately gives that Q is a regular presentation, using the fact that $\partial(A \cap B)$ is the disjoint union of $\partial A \cap B$, $A \cap \partial B$, $\partial A \cap \partial B$, for open convex cells A, B with $A \cap B \neq \emptyset$. Moreover $\mathscr{S}$ is full in Q. If $C \in Q$ is of the form $C = L_{(\sigma,a)} \cap E$, we write E as E_C. By definition each $C \in Q - \mathscr{S}$ is the intersection of E_C with a linear manifold, and if $C' < C$, $C' \in Q - \mathscr{S}$, $E_{C'} < E_C$ since $\mathscr{P}'$ is regular. Since $L_{(\sigma,a)}$ does not intersect any $(\leq q-1)$-dimensional face E of E_C with $E \cap \overline{S} = \emptyset$, by 5.4.5 $\dim L_{(\sigma,a)} \cap E_C \leq \dim E_{C-q}$. It remains to verify that if $C_1 \neq C_2, C_1, C_2 \in Q - \mathscr{S}$, then $E_{C_1} \neq E_{C_2}$. Let $C_1 = L_{(\sigma,a)} \cap E_{C_1}$, $C_2 = L_{(\tau,a)} \cap E_{C_2}$; $\sigma, \tau \in \mathfrak{a}$, $E_{C_1}, E_{C_2} \in \mathscr{P}' - \mathscr{S}$, $C_1 \neq \emptyset \neq C_2$. If $\sigma = \tau$, and $C_1 \neq C_2$, clearly $E_{C_1} \neq E_{C_2}$. If $\sigma \neq \tau$, then C_1 cannot be equal to C_2. In this case $E_{C_1} \subset \sigma\rho$, $E_{C_2} \subset \tau\rho$, (ρ defined in the first paragraph of the proof). But $\sigma\rho$ and $\tau\rho$ are disjoint, hence $E_{C_1} \neq E_{C_2}$. □

Remark. In the above proposition $\mathscr{P}'$ can be taken any presentation of $S * \Delta$ refining $\mathscr{P}$ and a join presentation of $S * \Delta$.

Proposition 5.4.10. *Let M be an $ND(n)$-space. Let $X \subset P$ be polyhedra such that $P = X \cup C$, C a closed convex cell, and $X \cap C = \partial C$, and* $\dim P = p \leq n$. *Let $f : P \to M$ be a map such that f/X is in general position. Then there exists an arbitrary close approximation g to f such that g is in general position and $g/X = f/X$.* 106

Proof. If $p = n$, any nondegenerate approximation of f would do. So let $p < n$. In particular $\dim C \leq p < n$.

Step A. Let D be an $(\leq n)$-dim-cell containing ∂C in its boundary, and such that

(1) $D = \partial C * \Delta$, Δ a closed convex $(n - p)$-cell

(2) $\Delta \cap C$ is a single point 'd' in the interior of both C and and Δ so that $C = d * \partial C$

(3) $D \cap P = C$.

This is clearly possible (upto polyhedral equivalence by considering $P \times 0$ in $V \times W$, (where V is the vector space containing P, W an $(n-p)$-dimensional vector space), and taking an $(n - p)$-cell Δ through $d \times 0$ in $d \times W$, for some $d \in C - \partial C$ etc. The join of the identity on ∂C and the retraction $\Delta \to d$ gives a retraction $r : D \to C$. Thus $(f/C) \cdot r$ is an extension of f/C. Since M is an $ND(n)$-space, $(f/C) \cdot r$ can be approximated by a non-degenerage map, say h, such that $h/\partial C = f/\partial C$. Let us patch up f/X and h, and let this be also called h; now h maps $X \cup D = P \cup D$ into M and is nondegenerate. Triangulate h so that the triangulation of $X \cup D$ with reference to which h is simplicial contains a subpresentation $\mathscr{D}$ which refines a join presentation of $\partial C * \Delta$ We apply 5.4.9 now, $\mathscr{D}$ will be $\mathscr{P}'$ there and we obtain, a point $a \in \Delta$, a 107
presentation $\mathscr{B}$ (what was called Q there) of $\partial C * a$. Each cell B of $\mathscr{B}$ not in ∂C, is the intersection of a unique E_B of $\mathscr{D}$ with a linear manifold, if $B' < B$ then $E_{B'} < E_B$ and $\dim B \leq \dim E_B - (n - p)$.

Step B. Let $B_1, \ldots, B_r$ be the elements of $\mathscr{B}$ not in ∂C, arranged so that $\dim B_i \leq \dim B_{i+1}$; for $1 \leq i < r$. Let $X_i = X \cup B_1 \cup \ldots \cup B_i$; X_i is a polyhedron. We define a sequence of embeddings $\mathscr{L}_i : X_i \to X \cup D$, such that

(1) $\mathscr{L}_i|X$ is the identity embedding of X in $X \cup D$

(2) $\mathscr{L}_i$ is an extension of $\mathscr{L}_{i-1}$

(3) $\mathscr{L}_i(B_i) \subset E_{B_i}$

(4) $h\mathscr{L}_i$ is in general position.

We shall construct the $\mathscr{L}_i$'s one at a time begining with $\mathscr{L}_0 : X \to X \cup D$, the inclusion, $h \cdot \mathscr{L}_0 = f/X$, is in general position, and we can start the induction.

Suppose $\mathscr{L}_{i-1}$ is already constructed. Then $\dim S_r(h\mathscr{L}_{i-1}) \leq d_r$; and by (2), (3), $\mathscr{L}_{i-1}$ embeds ∂B_i in ∂E_{B_i}; consider $h^{-1}(h\mathscr{L}_{i-1}(S_r(h\mathscr{L}_{i-1})))$ intersected with E_{B_i}. Since h is nondegenerate, these consists of a finite number of open convex cells of dimension $\leq d_r$. We apply 5.4.7 to this situation with $q = n - p$, $A = \overline{E}_{B_i}$, $S = \mathscr{L}_{i-1}(\partial B_i)$ and $\{B_1, \ldots, \}$ of 5.4.7 standing for the open cells of $h^{-1}(h\mathscr{L}_{i-1}(S_r(h\mathscr{L}_{i-1}))$ intersected with E_{B_i} for all $r \geq 1$. By 5.4.7, we can choose a point in E_{B_i} say e_i (a
108 of 5.4.7) so that $\mathscr{L}_{i-1}(\partial B_i) * e_i$ intersects all these (i.e. for all $r \geq 1$) in codimension $\geq (n-p)$. The join of $\mathscr{L}_{i-1}|\partial B_i$ and the map of a point b_i of B_i to e_i gives the required extension on B_i.

Then $\dim\{\mathscr{L}_i B_i \cap h^{-1}(h\mathscr{L}_{i-1}(S_r(h\mathscr{L}_{i-1}))\} \leq d_r - (n-p)$, equivalently $\dim\{(h\mathscr{L}_i B_i) \cap h\mathscr{L}_{i-1}(S_r(h\mathscr{L}_{i-1}))\} d_{r+1} \leq d_{r+1}$, that is $\dim\{(h\mathscr{L}_i B_i) \cap h\mathscr{L}_i(S_r(h\mathscr{L}_i|X_{i-1}))\} d_{r+1}$, since $h\mathscr{L}_i$ is an extension of $h\mathscr{L}_{i-1}$.

Since

$$\begin{aligned} S_{r+1}(h\mathscr{L}_i) = S_{r+1}(h\mathscr{L}_{i-1}) & \\ & \cup \{B_i \cap (h\mathscr{L}_i)^{-1}(h\mathscr{L}_i(S_r(h\mathscr{L}_i|X_{i-1}))) \\ & \cup \{S_r(h\mathscr{L}_i|X_{i-1}) \cap (h\mathscr{L}_i)^{-1}(h\mathscr{L}_i)(B_i)\} \end{aligned}$$

and since $h\mathscr{L}_{i-1}$ is already in general position, $\dim S_{r+1}(h\mathscr{L}_i) \leq d_{r+1}$. At the last stage, we get an imbedding $\mathscr{L}_r$ of $X \cup \partial C * a$ in $X \cup D$, such that $h\mathscr{L}_r$ is an general position.

That $h\mathscr{L}_r$ can be chosen as close to f as we like is clear. □

Theorem 5.4.11. *Let M be an $ND(n)$-space, $X \subset P$ polyhedra* $\dim p \leq n$ *and $f : P \to M$ a map such that $f|X$ is in general position. Then there exists an arbitrary close approximation g to f such that $g|X = f|X$, and g is in general position.*

Proof. Let $\mathscr{P}$ be a regular presentation of P with X covered by a sub-presentation $\mathscr{X}$. Let $(\mathscr{P} - \mathscr{X}) = \{A_1, \ldots, A_r\}$ be arranged so that 109
$\dim A_i \leq \dim A_{i+1}$, $1 \leq i < r$. Let $P_i = X \cup A_1 \cup \ldots \cup A_i$, $X_i = P_{i-1}$. Apply proposition 5.4.10 successively to $(P_1, X_1) \ldots, (P_r, X_r)$.

This requires the following comment: We must use our approximation theorem, which for M and $\epsilon > 0$ gives $\delta(\epsilon) > 0$, such that for any $Y \supset Z$, $h_1 : Y \to M$, $h_2 : Z \to M$, if h_2 is polyhedral, and $h_1|Z$ is a $\delta(\epsilon)$-approximation to h_2, then there is $h_3 : Y \to M$, a polyhedral extension of h_2, which is an ϵ-approximation to h_1.

We want g to be an ϵ-approximation to f.

Define $\epsilon_r = \epsilon, \epsilon_{i-1} = \delta\left(\dfrac{\epsilon_i}{2}\right)$.

Denote $f|P_i$ by f_i. We start with $g_0 = f_0 = f|X$. Suppose g_{i-1} is defined on P_{i-1} such that g_{i-1} is in general position and is an ϵ_{i-1} approximation to f_{i-1}. Then we first extend g_{i-1} to P_i say f_i' so that f_i' is an $\epsilon_{\frac{i}{2}}$ approximation to f_i (this is possible since $\epsilon_{i-1} = \delta(\epsilon i/2)$) by the approximation theorem. Then we use 5.4.10 to get an $\epsilon_{i/2}$ approximation g_i to f_i' such that g_i is in general position and $g_i|P_{i-1} = g_{i-1}$. g_i is an ϵ_i-approximation to f_i and is in general position. g_r gives the required extension. □

By the methods of 5.4.10, the following proposition can be proved:

Proposition 5.4.12. *Let M be $ND(n)$;* $\dim p \leq n$*, $P = X \cup C$, where C is a closed p-cell, $X \cap C = \partial C$. Let $f : P \to M$ be a map, such that $f|X$* 110
is nondegenerate; and call $\dim X = x$*. Then there is a nondegenerate approximation $g : P \to M$, arbitrarily close to f, such that $g|X = f|X$, and $S(g) = S(f|X)$ plus (a finite number of open convex cells of* $\dim \leq$ $\mathrm{Max}(2p - n, p + x - n)$*)*

Sketch of the proof: First we proceed as in Step A of 5.4.10. Now $p = \dim C$. In Step B) instead of 4) we write

$$\dim\{B_i \cap (h\mathscr{L}_i)^{-1}(h\mathscr{L}_{i-1}(X_{i-1}))\} \leq \mathrm{Max}(2p - n, p + x - n).$$

And in the proof instead of the mess before, we have only to bother about $h^{-1}(h\mathscr{L}_{i-1}(X_{i-1}))$, intersected with E_{B_i}.

$$h^{-1}(h\mathscr{L}_{i-1}(X_{i-1})) = h^{-1}(h\mathscr{L}_{i-1}(X)) \cup h^{-1}(h\mathscr{L}_{i-1}(B_i \cup \ldots \cup B_{i-1}) = \\ = h^{-1}(f(X)) \cup h^{-1}(h\mathscr{L}_{i-1}(B_1 \cup \ldots \cup B_{i-1})).$$

Now the only possibility of $h^{-1}(f(X))$ intersecting E_{B_i} is when $E_{B_i} \subset h^{-1}(f(X))$ since h is simplicial. Since $\dim B_i \leq \dim E_{B_i} - (n-p)$, it already intersects in the right codimension. And the intersections with second set can be made minimal as before. ◻

Theorem 5.4.13. *Let M be $ND(n)$; $X \supset P$; $\dim p \leq n$, $f : P \to M$ a map such that $f|X$ is an imbedding. Then arbitrary close to f is a map $g : P \to M$, such that $g|X = f|X$ and calling $x = \dim X$, $p = \dim P - X$,*

$$\dim S(g) \leq \mathrm{Max}(2p-n, p+x-n).$$

Proof. This follows from 5.4.12, as 5.4.11 from 5.4.10. ◻

111 This theorem is useful in proving the following embedding theorem for $ND(n)$-spaces.

Theorem 5.4.14 (Stated without proof)**.** *Let M be a $ND(n)$-space, P a polyhedron of dimension $p \leq n-3$ and $f : P \to M$ a $(2p-n+1)$-connected map. Then there is a polyhedron Q in M and a simple homotopy equivalence $g : P \to Q$ such that the diagram*

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[dr, "f"'] & Q \arrow[d, "\text{inclusion}"] \\ & M \end{tikzcd}$$

is homotopy commutative.

The method of Step A in 5.4.10, gives;

Proposition 5.4.15. *Let M be an $ND(n)$-space, and P a polyhedron of dimension $p \leq n$ and $f : P \to M$ be any map. Then $\exists$ a regular presentation $\mathscr{P}$ of P, simplicial presentation $\mathscr{M}$ of M and an arbitrary close approximation g to f such that, for each $C \in \mathscr{P}$, $g|(C)$ is a linear embedding, and $g(C)$ is contained in a simplex σ_C of $\mathscr{M}$ of dimension $=$ dimension $C + (n - p)$ and $g(\partial C) = \partial(gC) \subset \partial\sigma_C$. Moreover $\mathscr{M}$ can be assumed to refine a given regular presentation of M.*

Also a relative version of 5.4.15 could be obtained. □

And from this and 5.4.13.

Theorem 5.4.16. *Let $f : P \to M$ be a map from a polyhedra P of* dim $= p$ *into an $ND(n)$-space M, $p \leq n$, and let Y be a subpolyhedron of M of dimension y. Then there exists an arbitrary close approximation g to f such that*

$$\dim(g(P) \cap Y) \leq p + y - n.$$

And a relative version of 5.4.16. □ **112**

5.4.17 It should be remarked that the definition of 'general position' in 5.4.1 is a definition of general position, and other definitions are possible, and theorems, such as above can be proved. Here we formulate another definition and a theorem which can be proved by the methods of 5.4.10.

A dimensional function $d : P \to \{0, 1, \ldots\}$ is a function defined on a polyhedron, with non-negative integer values, such that there is some regular presentation $\mathscr{P}$ of P such that for all $C \in \mathscr{P}$, $x \in C$, $d(x) \geq \dim C$, and d is constant on C.

We say $d_1 \leq d_2$, if for all $x \in P$, $d_1(x) \leq d_2(x)$.

If $f : P \to M$ is a nondegenerate map, and d a dimensional function, and $k_1, \ldots, k_s$ non-negative integers, we define

$$\begin{aligned} S\,d(f; k_1, \ldots, k_s) = f^{-1}\{m \in M | \exists \text{ distinct points}\\ x_1, \ldots, x_s \in P, \text{ such that}\\ d(x_i) \leq k_i, \text{ and } f(x_i) = m \text{ for all } i\}. \end{aligned}$$

It is possible that such a set is a union of open simplexes, and hence its dimension is easily defined.

A map $f : P \to M$ is said to be *n-regular* with reference to a dimensional function d on P if it is nondegenerate and

$$\dim S d(f; k_1, \ldots, k_s) \leq k_1 + \cdots + k_s - (s-1)n.$$

for all s, and all s-tuples of non-negative integers.

113 If $\dim P \leq n$, and since we have f nondegenerate then it is possible to show that a map f is n-regular if it satisfies only a finite number of such inequalities, namely those for which all $k_i \leq n-1$ and $s < 2n$.

The theorem that can be proved is this;

Theorem . *Let $X \subset P$, $f : P \to M$, where M is $ND(n)$ and $\dim P \leq n$. Let d_X and d_P be dimensional functions on X and P, with $d_X \leq d_P|X$. Suppose $f|X$ in n-regular with reference to d_X. Then f can be approximated arbitrarily closely by $g : P \to M$ with $g|X = f|X$ and g n-regular with reference to d_P.*

The proof is along the lines of theorem 5.4.11. We find a regular presentation $\mathscr{P}$ of P with a subpresentation covering X, and such that d_X and d_P are constant on elements of $\mathscr{P}$. We utilise theorem 5.4.10 to get g on the cells of $\mathscr{P}$ one at a time; in the final atomic construction, analogous to part (B) of 5.4.10, we will have

$$S \subset \partial E$$

where S is a $(k-1)$-sphere. E a cell of dimension $\geq k+q$, where $q = n-p$ (the cell we are extending over is a p-cell, on which d_P is constant $\geq p$). We have to insert a k-cell that will intersect such things as

$$h^{-1}(S_{d_p}(\phi_{i-1}; k_1, \ldots, k_s))$$

in dimension

$$\begin{gathered}\dim S_{d_p}(\phi_{i-1}; k_1, \ldots, k_s) - q \\ \leq k_1 + \cdots + k_s + d_P(p\text{-cell}) - \text{s.n.}\end{gathered}$$

114 We can do this for our situation; this inequality will imply g_i is n-regular. □

Finally we can define on any polyhedron P a canonial dimensional function d:

$$d(x) : \mathrm{Min}\Big\{ \dim(\text{Stary in [Star of } x \text{ in } P]$$

$$y \in \quad [\text{Star of } x \text{ in } p]\Big\}$$

A function n-regular with reference to this d will be termed, perhaps, in general position, it being understood that the target of the function is $ND(n)$. Thus:

Corollary . *If $X \subset P$, $\dim P \leq n$, $f : P \to M$, M a $ND(n)$-space, and if $f|X$ is in general position then $f|X$ can be extended to a map $g : P \to M$ in general position such that g closely approximates f.*

Conclusion 5.4.18. Finally, it should be remarked, that the above 'general position' theorems, interesting though they are; are not delicate enough for many applications in manifolds. For example, one need: If $f : X \to M$ a map of a polyhedron X into a manifold, and $Y \subset M$, the approximation g should be such that not only $\dim(g(X) \cap Y)$ is minimal, but also should have $\overline{S_r(g)}$ intersect Y minimally e.g. if $2x+y < 2n$, $\overline{S(g)}$ should not intersect Y at all. The above procedure does not seem to give such results. If for example we know that Y can be moved by an isotopy of M to make its intersections minimal with some subpolyhedra of M, then these delicate theorems can be proved. This is true in the case of manifolds, and we refer to Zeeman's notes for all those theorems.

Chapter 6

Regular Neighbourhoods

The theory of regular neighbourhoods in due to J.H.C. Whitehead, and **115**
it has proved to be a very important tool in the study of piecewise linear manifolds. Some of the important features of regular neighbourhoods, which have proved to be useful in practice can be stated roughly as follows:

(1) a second derived neighbourhood is regular (2) equivalence of two regular neighbourhoods of the same polyhedron (3) a regular neighbourhood collapses to the polyhedron to which it is regular neighbourhood (4) a regular neighbourhood can be characterised in terms of collapsing. Whitehead's theory as well as its improvement by Zeeman are stated only for manifolds. Here we try to obtain a workable theory of regular neighbourhoods in arbitrary polyhedra; our point of view was suggested by M. Cohen.

If X is a subpolyhedron of a polyhedron K, we define a regular neighbourhood of X in P to be any subpolyhedron of K which is the image of second derived neighbourhood of X, under a polyhedral equivalence of K which is fixed on X. It turns out that this is a polyhedral invariant, and any two regular neighbourhoods of X in K are equivalent by an isotopy which fixed both X and the complement of a common neighbourhood of the two regular neighbourhoods. To secure (4) above, we introduce "homogeneous collapsing". Applications to manifolds are scattered over the chapter. These and similar theorems are due espe- **116**
cially to Newman, Alexander, Whitehead and Zeeman.

6.1 Isotopy

Let X be a polyhedron and I the standard 1-cell.

Definition 6.1.1. An *isotopy* of X in itself is a polyhedral self-equivalence of $X \times I$, which preserves the I-coordinate.

That is, if h is the polyhedral equivalence of $X \times I$, writing $h(x,t) = (h_1(x,t), h_2(x,t))$, we have $h_2(x,t) = t$. The map of X into itself which takes x to $h_1(x,t)$ is a polyhedral equivalence of X and we denote this by h_t. Thus we can write h as

$$h(x,t) = (h_t(x), t).$$

We usually say that 'h is an isotopy between h_0 and h_1', or 'h_0 is isotopic to h_1' or 'h is an isotopy from h_0 to h_1'. The composition (as functions) of two isotopics is again an isotopy, and the composition of two functions isotopic to identity is again isotopic to identity.

Now we describe a way of constructing isotopies, which is particularly useful in the theory of regular neighbourhoods.

Proposition 6.1.2. *Let X be the cone on A. Let $f : X \to X$ be a polyhedral equivalence, such that $f|A = \mathrm{id}_A$. Then there is an isotopy $h : X \times I \to X \times I$, such that $h|(X \times 0) \cup A \times I =$ identity and $h_1 = f$.*

Proof. Let X be the cone on A with vertex v, the interval $I = [1, 0]$ is the cone on 1 with vertex 0. Therefore by 4.3.19, $X \times I$ is the cone on $X \times 1 \cup A \times I$ with vertex $(v, 0)$. Define

$$h : X \times 1 \cup A \times I \to X \times 1 \cup A \times I$$

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$$\begin{aligned} h(x,1) &= (f(x), 1) \quad \text{for} \quad x \in X \\ h(a,t) &= (a,t) \quad \text{for} \quad a \in A, t \in I. \end{aligned}$$

Since $h|A = \mathrm{id}_A$, h is well defined and is clearly a polyhedral equivalence. We have h defined on the base of the cone; we extend it radially, by mapping $(v, 0)$ to $(v, 0)$, that is we take the join of h and

Identity on $(v, 0)$. Calling this extension also h, we see that h is a polyhedral equivalence and is the identity on $(A \times I) * (v, 0)$. Since $X \times 0 \cup A \times I \subset (A \times I) * (v, 0)$, h is identity on $X \times 0 \cup A \times I$. To show that h preserves the I-coordinate, it is enough to check on $(X \times 1) * (v, 0)$, and this can be seen for example by observing that the $t(x, 1) + (1-t)(v, 0)$ of $X \times I$ with reference to the conical representation is the same as the point $(tx + (1-t)v, t)$ of $X \times I$ with reference to the product representation, and writing down the maps. □

If h is an isotopy of X in itself, $A \subset X$, and if $h|A \times I = \mathrm{Id}(A \times I)$ as in the above case, we say that *h leaves A fixed.* And some times, if h is an isotopy between Id_X and h_1, we will just say that *'h is an isotopy of X'*, and then an arbitrary isotopy will be referred to as 'an isotopy of X in itself'. Probably this is not strictly abhered to in what follows; perhaps it will be clear from the context, which is which.

From the above proposition, the following well known theorem of Alexander can be deduced:

Corollary 6.1.3. *A polyhedral automorphism of an n-cell which is the identity on the boundary, is isotopic to the identity by an isotopy leaving the boundary fixed.* **118**

It should be remarked that we are dealing with I-isotopics and these can be generalised as follows:

Definition 6.1.4. Let J be the cone on K with vertex 0. A J-isotopy of X is a polyhedral equivalence of $X \times J$ which preserves the J-coordinate.

The isotopy is said to be between the map $X \times 0 \to X \times 0$ and the map $X \times K \to X \times K$, both induced by the equivalence of $X \times J$. And we can prove as above:

Proposition 6.1.5. *Let X be the cone on A, and let $f : X \times K \to X \times K$ be a polyhedral equivalence preserving the K-coordinate and such that $f|A \times K = \mathrm{Id}_{A \times K}$. Then f is isotopic to the identity map of X by a J-isotopy $h : X \times J \to X \times J$, such that on $A \times J$ and $X \times 0$, h is the identity map.*

This is in particular applicable when J is an n-cell.

6.2 Centerings, Isotopies and Neighbourhoods of Subpolyhedra

Let $\mathscr{P}$ be a regular presentation of a polyhedron P, and let η, θ be two centerings of $\mathscr{P}$. Then obviously the correspondence

$$\eta C \Longleftrightarrow \theta C, \quad C \in \mathscr{P}$$

gives a simplicial isomorphism of $d(\mathscr{P}, \eta)$ and $d(\mathscr{P}, \theta)$, which gives a polyhedral equivalence of P. We denote this by $f_{\theta,\eta}$ (coming from map $\eta C \to \theta C$). Clearly $f_{\eta,\theta} = (f_{\theta,\eta})^{-1}$, and $f_{\eta,\theta} 0 f_{\theta,\zeta} = f_{\eta,\zeta}$ where η, θ, ζ are
119 three centerings of

Proposition 6.2.1. *The map $f_{\theta,\eta}$ described above is isotopic to the identity through an isotopy $h : P \times I \to P \times I$, such that if for a $C \in \mathscr{P}$, η and θ are the same on C and all $D \in \mathscr{P}$ with $D < C$ then $h|\overline{C} \times I$ is identity.*

Proof. First, let us consider the case when η and θ differ only on a single cell A. Then $f_{\theta,\eta}$ is identity except on $|St(\eta A, d(\mathscr{P}, \eta))| = |St(\theta A, d(\mathscr{P}, \theta))|$. This is a cone, and $f_{\theta,\eta}$ is identity on its base; then by 6.1.2. We obtain an isotopy of $|St(\eta A, d(\mathscr{P}, \eta))|$, which fixes the base. Hence it will patch up with the identity isotopy of $\overline{K - |(St(\eta A, d(\mathscr{P}, \eta)))|}$.

The general $f_{\theta,\eta}$ is the composition of finitely many of these special cases, and we just compose the isotopies obtained as above in the special cases. For isotopies constructed this way, the second assertion is obvious. ◻

Let X be a subpolyhedron of a polyhedron P, and let $\mathscr{P}$ be a simplicial presentation of P containing a full subpresentation $\mathscr{X}$ covering X. We have defined $N_{\mathscr{P}}(\mathscr{X})$ (in 3.1) as the full subpresentation of $d\mathscr{P}$, whose vertices are ηC for $C \in \mathscr{P}$ with $\overline{C} \cap X \neq \emptyset$. This of course depends on a centering η of $\mathscr{P}$, and to make this explicit we denote it by $N_{\mathscr{P}}(\mathscr{X}, \eta)$. $|N_{\mathscr{P}}(\mathscr{X}, \eta)|$ is usually called a '*second derived neighbourhood of X*'. We know that $|N_{\mathscr{P}}(\mathscr{X}, \eta)|$ is a neighbourhood of X, and that X is a deformation retract of $|N_{\mathscr{P}}(\mathscr{X}, \eta)|$ (see 3.1). Our next aim is to show that any two second derived neighbourhoods of X in P are equiva-
120 lent by an isotopy of P leaving X, and a complement of a neighbourhood of both fixed. We go through a few preliminaries first.

Ex. 6.2.2. With the same notation as above. Let η and θ be two centerings of $\mathscr{P}$ such that for every $C \in \mathscr{P} - \mathscr{X}$, with $\overline{C} \cap X \neq \emptyset$, $\eta C = \theta C$. Then

$$|N_{\mathscr{P}}(\mathscr{X}, \eta)| = |N_{\mathscr{P}}(\mathscr{X}, \theta)|.$$

[Hint: This can be seen for example by taking subdivisions of $\mathscr{P}$ which are almost the same as $d(\mathscr{P}, \eta)$ and $d(\mathscr{P}, \theta)$, but leave $\mathscr{X}$ unaltered].

Proposition 6.2.3. *With X, P, $\mathscr{X}$, $\mathscr{P}$ as above, let η and θ be two centerings of $\mathscr{P}$, and $\mathcal{U}$ the union of all elements of $\mathscr{P}$, whose closure intersects X. Then there is an isotopy h of P fixed on X and $P - U$, such that $h_1(|N_{\mathscr{P}}(\mathscr{X}, \eta)|) = |N_{\mathscr{P}}(\mathscr{X}, \theta)|$.*

Proof. We first observe that $P - U$ is a subpolyhedron of P and there is a full subpresentation $\mathfrak{a}$ of $\mathscr{P}$ which covers $P - U$, namely, $C \in \mathfrak{a}$ if and only if $\overline{C} \cap X = \emptyset$. By 6.2.2 we can change η and θ on $\mathscr{X}$ and $\mathfrak{a}$ without altering $|N_{\mathscr{P}}(\mathscr{X}, \eta)|$ and $|N_{\mathscr{P}}(\mathscr{X}, \theta)|$. So we may assume that η and θ are the same on $\mathscr{X}$ and $\mathfrak{a}$. The isotopy h of proposition 6.2.1 with the new $f_{\theta,\eta}$ in the hypothesis has the desired properties. □

With X, P, $\mathscr{X}$, $\mathscr{P}$ as above, let $\varphi : P \to [0, 1]$ be map given by: if v is a $\mathscr{X}$-vertex $\varphi(v) = 0$, if v is a $(\mathscr{P} - \mathscr{X})$-vertex $\varphi(v) = 1$, and φ is linear on the closures of $\mathscr{P}$-simplexes. Then $\varphi^{-1}(0) = X$, since $\mathscr{X}$ is full in $\mathscr{P}$. If σ is a simplex of $\mathscr{P} - \mathscr{X}$, then $\overline{\sigma} \cap X \neq \emptyset$, if and only if $\varphi(\sigma) = (0, 1)$. If σ is a simplex of $\mathfrak{a}$ ($\mathfrak{a}$ as in the proof of proposition 2.3) **121**
then $\varphi(\sigma) = 1$. Roughly, the map φ ignores the parts of P away from X and focusses its attention on a neighbourhood of X. We will use this map often.

Proposition 6.2.4. *With the above hypotheses, if $0 < \alpha < \beta < \gamma < 1$, then there is an isotopy h of P, taking $\varphi^{-1}([0,\beta])$ onto $\varphi^{-1}([0,\alpha])$ and leaving X and $P - \varphi^{-1}([0,\gamma])$ fixed.*

Proof. Let φ be the map: $P \to [0, 1]$ described above. Choose a centering ζ of $\mathscr{P}$ as follows: if σ is a simplex of $\mathscr{P}$ with $\overline{\sigma} \cap X \neq \emptyset$, then $\varphi(\zeta\sigma) = \gamma$, and choose ζ arbitrarily on $\mathscr{X}$ and $\mathfrak{a}$. Let $\mathscr{P}'$ denote $d(\mathscr{P}, \zeta)$. Let $\mathscr{X}'$ be the subpresentation with $|\mathscr{X}'| = |\mathscr{X}| = X$. Obviously $|N_{\mathscr{P}}(\mathscr{X}, \zeta)| = \varphi^{-1}([0, \gamma])$, and if ρ is any simplex of $\mathscr{P}'$ with vertices both in and out of $\mathscr{X}'$, then $\varphi(\rho) = (0, \lambda)$. Now choose two centerings θ and η of $\mathscr{P}'$ such that if $\rho \in \mathscr{P}'$ and $\varphi(\mathscr{P}) = (0, \lambda)$, then $\varphi(\eta C) = \beta$ and $\varphi(\theta C) = \mathcal{L}$ and arbitrarily otherwise. Then clearly

$$
\begin{aligned}
& |N\mathscr{P}'(\mathscr{X}', \eta)| = \varphi^{-1}([0, \beta]) \\
\text{and} \quad & |N\mathscr{P}'(\mathscr{X}', \theta)| = \varphi^{-1}([0, \lambda]).
\end{aligned}
$$

We apply 6.2.3 now, and $\mathscr{U}$ of 6.2.3 in this case happens to be $\varphi^{-1}([0, \lambda])$. □

Proposition 6.2.5. *Let $\mathscr{P}$ be a simplicial presentation, $\mathscr{X}$ a full subpresentation of $\mathscr{P}$, $|\mathscr{P}| = P$, $|\mathscr{X}| = X$, η a centering of $\mathscr{P}$; and*
122 *$N = |N_{\mathscr{P}}(\mathscr{X}, \eta)|$. Let φ be a simplicial refinement of $d(\mathscr{P}, \eta)$ with $\mathscr{Y}$ the subpresentation covering X; let θ a centering of $\mathscr{O}$ and $N' = |N_Q(\mathscr{Y}, \theta)|$. Finally, let $\mathscr{U}$ be a neighbourhood of N. Then there is an isotopy h of P, taking N onto N'; and leaving $P - \mathscr{U}$ and X fixed.*

Remark. Note that if $\mathscr{U}$ were somewhat large, or if there were no $\mathcal{U}$ in the statement, then the proposition is an immediate consequence of 6.2.3 and 6.2.4.

Proof. We first replace the centering η by a centering η' as follows: Let $\varphi : P \to [0, 1]$ be the usual function given by, φ ($\mathscr{X}$-vertex) $= 0$, $\varphi(\mathscr{P} - \mathscr{X})$-vertex $= 1$, and φ is linear on the closures of $\mathscr{P}$-simplexes. Choose η' such that if ρ is a simplex of $\mathscr{P}$ with $\varphi(\rho) = (0, 1)$, then $\varphi(\eta'\rho) = \frac{1}{2}$, $f\eta'$, η is a polyhedral equivalence carrying $|N_{\mathscr{P}}(\mathscr{X}, \eta)|$ onto $|N_{\mathscr{P}}(\mathscr{X}, \eta')| = \varphi^{-1}([0, \frac{1}{2}])$. Actually $f_{\eta', \eta}$ is isotopic to the identity, but we will need only that it is a polyhedral equivalence. Let $f_{\eta', \eta}(\mathscr{U}) = \mathscr{U}'$. As $\mathscr{U}'$ is a neighbourhood of $\varphi^{-1}([0, \frac{1}{2}])$; we can find a $\gamma > \frac{1}{2}$ such that $\varphi^{-1}([0, \gamma]) \subset \mathscr{U}'$. Since $f_{\eta', \eta}$ is simplicial with reference to $d(\mathscr{P}, \eta)$ and $d(\mathscr{P}, \eta')$ and Q is refinement of $d(\mathscr{P}, \eta)$, $f_{\eta', \eta}$ carries Q onto a refinement of $d(\mathscr{P}, \eta')$. Let us call this Q', similarly $f_{\eta', \eta}(\mathscr{Y})$ by $\mathscr{Y}' \cdot |\mathscr{Y}'| = X$. Let the centering of Q' induced from θ be θ'. We have,

$$
f_{\eta', \eta}(|N_{\mathscr{P}}(\mathscr{X}, \eta)|) = |N_{\mathscr{P}}(\mathscr{X}, \eta')| = \varphi^{-1}([0, \frac{1}{2}])
$$

and

$$
f_{\eta', \eta}(|N_Q(\mathscr{Y}, \theta)|) = |N_{Q'}(\mathscr{Y}', \theta')|.
$$

123 Now choose another centering θ_1 of Q' as follows:

Let $\mathcal{L}$, $(0 < \mathcal{L} < \frac{1}{2})$ be such that if v is a vertex of Q' not in X, then $\varphi(v) > \mathcal{L} \cdot \theta_1$ is chosen so that if $\sigma \in Q'$ has vertices in and out of X, then $\varphi(\theta_1\sigma) = \mathcal{L}$. Then clearly $|N_{Q'}(\mathscr{Y}', \theta_1)| = \varphi^{-1}([0, \mathcal{L}])$. By 6.2.4, there is an isotopy h of P, leaving X and complement of $\varphi^{-1}([0,\gamma])$ fixed, with h_1 taking $\varphi^{-1}([0, \frac{1}{2}])$ onto $\varphi^{-1}([0, \mathcal{L}])$. By 6.2.3 there is an isotopy h' leaving X and complement of $\varphi^{-1}([0, \frac{1}{2}])$ fixed, with h'_1 taking $\varphi^{-1}([0, \mathcal{L}]) = |N_{Q'}(\mathscr{Y}', \theta_1)|$ onto $|N_{Q'}(\mathscr{Y}', \theta')|$.

Let $\tilde{f}_{\eta',\eta}$ be the isotopy of P in itself given by $\tilde{f}_{\eta',\eta}(p,t) = (f_{\eta',\eta}(p), t)_p P$. Then $g = \tilde{f}^{-1}_{\eta',\eta} \circ h' \circ h \circ \tilde{f}_{\eta',\eta}$ is the required isotopy. First $g_1 = f_{\eta',\eta^{-1}} \circ h'_1 \circ h_1 \circ f_{\eta',\eta}$ carries N onto N'. Secondly since $P - \mathscr{U}' \subset P - \varphi^{-1}([0, \frac{1}{2}])$ and $P - \mathscr{U}' \subset P - \varphi^{-1}([0,\gamma])$, h and h' are fixed on $P - \mathscr{U}'$. They are also fixed on X. As $f_{\eta',\eta}$ carries X onto X, $\mathscr{U}$ onto $\mathscr{U}'g$ also fixes X and $P - \mathscr{U}$. ◻

Corollary 6.2.6. *Let X be a subpolyhedron of a polyhedron P. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two simplicial presentation of P, containing full subpresentations $\mathscr{X}_1$ and $\mathscr{X}_2$ respectively with $|\mathscr{X}_1| = |\mathscr{X}_2| = X$. Let θ_1 and θ_2 be centering of $\mathscr{P}_1$ and $\mathscr{P}_2$, and $N_1 = |N_{\mathscr{P}_1}(\mathscr{X}_1, \theta_1)|$, $N_2 = |N_{\mathscr{P}_2}(\mathscr{X}_2, \theta_2)|$, and $\mathscr{U}$ a neighbourhood of $N_1 \cup N_2$ in P. Then there is an isotopy of P leaving X and $P - \mathscr{U}$ fixed and taking N_1 onto N_2.*

Proof. Take a common subdivision Q of $d(\mathscr{P}_1, \theta_1)$ and $d(\mathscr{P}_2, \theta_2)$ and 124
apply 6.2.5 twice. ◻

6.3 Definition of "Regular Neighbourhoods"

Let X be a subpolyhedron of a polyhedron P.

Definition 6.3.1. A subpolyhedron N is said to be regular neighbourhood of X in P if there is a polyhedral equivalence h of P on itself, leaving X fixed, such that $h(N)$ is a second derived neighbourhood of X.

More precisely N is a regular neighbourhood of X if and only if

(i) there is a simplicial presentation $\mathscr{P}$ of P with a full subpresentation $\mathscr{X}$ covering X and a centering η of $\mathscr{P}$; and

(ii) a polyhedral equivalence h of P fixed on X such that $h(N) = |N_{\mathscr{P}}(\mathscr{X}, \eta)|$.

Regular neighbourhoods do exist and if N is a regular neighbourhood of X in P, then N is a neighbourhood of X in P.

Proposition 6.3.2. *If N_1 and N_2 are two regular neighbourhoods of X in P and $\mathscr{U}$ a neighbourhood of $N_1 \cup N_2$ in P, then there exists in isotopy h of P taking N_1 onto N_2 and leaving X and $P - \mathscr{U}$ fixed.*

Proof. Let $\mathscr{P}_i$, $\mathscr{X}_i$, η_i, h_i, $i = 1, 2$, be such that

$$h_i |N_{\mathscr{P}_i}(\mathscr{X}_i, \eta_i)| = N_i, i = 1, 2.$$

Let Q_1 be a subdivision of $d(\mathscr{P}_1, \eta_1)$ such that h_1 is simplicial with
125 reference to Q_1, and let $\mathscr{Y}_1$ be the subpresentation of Q_1 covering X. Let θ_1 be a centering of Q_1.

$$h_1 |N_{Q_1}(\mathscr{Y}_1, \theta_1)| = |N h_1 Q_1(\mathscr{Y}_1, h_1\theta_1)| = N_1'$$

say [Note that h_1 is fixed on X].

By 6.2.5, there is an isotopy f, fixed on X and $P - h_1^{-1}(\mathscr{U})$ with f_1 taking $|(N_{\mathscr{P}_1}(\mathscr{X}_1, \eta_1)|$ onto $|N_{Q_1}(\mathscr{Y}_1, \theta_1)|$. Then $\tilde{h}_1 f \tilde{h}_1^{-1}$ is an isotopy of P fixed on X and $P - \mathscr{U}$ and $(\tilde{h}_1 f \tilde{h}_1^{-1})_1 = h_1 f_1 h_1^{-1}$ takes N_1 onto N_1' (where $\tilde{h}_1$ is the isotopy of P in itself given by $\tilde{h}_1(p, t) = (h_1(p), t)$). Working similarly with $\mathscr{P}_2$, we obtain f' with $\tilde{h}_2 f' \tilde{h}_2^{-1}$ fixed on X and $P - \mathscr{U}$ and $(\tilde{h}_2 f' \tilde{h}_2^{-1})_1 = h_2 f_1' h_2^{-1}$ taking N_2 onto N_2'. Now N_1' and N_2' are genuine second derived neighbourhoods, and $\mathscr{U}$ is a neighbourhood of $N_1' \cup N_2'$. Hence by 6.2.6 there is an isotopy g of P fixed on X and $P - \mathscr{U}$, with $g_1(N_1') = N_2'$.

$(\tilde{h}_2 f' \tilde{h}_2^{-1})^{-1} g(\tilde{h}_1 f \tilde{h}_1^{-1})$ is the required isotopy. □

Proposition 6.3.3. *If $f : P \to P'$ is a polyhedral equivalence and N a regular neighbourhood of X in P, then $f(N)$ is a regular neighbourhood of $f(X)$ in P'.*

Proof. Let $\mathscr{P}$, $\mathscr{P}'$ be a simplicial presentations of P and P' with reference to which f is simplicial, η be a centering of $\mathscr{P}$, $f(\eta) = \eta'$ the

induced centering on $\mathscr{P}'$. We can assume that $\mathscr{P}$, $\mathscr{P}'$ contain full subpresentations $\mathscr{X}$, $\mathscr{X}'$ covering X and X'; (by going to subdivisions if necessary). $f(|N_{\mathscr{P}}(\mathscr{X},\eta)|) = |N_{\mathscr{P}'}(\mathscr{X}',\eta')|$. By definition, there is a polyhedral equivalence $\mathfrak{a}$ of P fixed on X such that $p(N) = |N_{\mathscr{P}}(\mathscr{X},\eta)|$. **126**
Let p' be the polyhedral equivalence of P' given by $f \circ p \circ f^{-1}$. Then $(f \circ p \circ f^{-1})(N') = (f \circ p)(N) = f(|N_{\mathscr{P}}(\mathscr{X},\eta)|) = |N_{\mathscr{P}'}(\mathscr{X}',\eta')|$, and if $x' \in f(X)$, $f^{-1}(x') \in X$, therefore $f \circ p \circ f^{-1}(x') = f \circ p(f^{-1}(x')) = f \circ f^{-1}(x') = x'$. □

Notation 6.3.4. If A is a subset of a polyhedron P, we will denote by $\mathrm{int}_P N$ and $bd_P N$ the interior and the boundary of N in the (unique) topology of P.

Ex. 6.3.6. If N is a regular neighbourhood of X in P, and $B = bd_P N$, then $X \subset N - B$.

Ex. 6.3.7. Let $X \subset N \subset Q \subset P$ be polyhedra, with $N \subset \mathrm{int}_P Q$. Then N is a regular neighbourhood of X in Q if and only if N is a regular neighbourhood of X in P.

Ex. 6.3.8. Let $X \subset P$ be polyhedra. If A is any subpolyhedron of P, let A' denote the polyhedron $A - \mathrm{int}_P X$. Then N is a regular neighbourhood of X in P if and only if N' is a regular neighbourhood of X' in P'.

Ex. 6.3.9. Let A be any polyhedron, and I the standard 1-cell. Let $0 < \alpha < \beta < \gamma < 1$ be three numbers. Then, $A \times [0, \mathcal{L}]$ is a regular neighbourhood of A in $A \times I$, and $A \times [\mathcal{L}, \gamma]$ is a regular neighbourhood of $A \times \beta$ in $A \times I$.

6.3.10 Notation and proposition

If $\mathscr{P}$ is any simplicial presentation and $\sum$ any set of vertices of $\mathscr{P}$, we denote by $\mathscr{P}_{\sum}$ the maximal subpresentation of $\mathscr{P}$ whose set of vertices is $\sum \cdot \mathscr{P}_{\sum}$ if full in $\mathscr{P}$. We write $\delta_{\mathscr{P}}(\sum)$ or $\delta(\sum)$ (when $\mathscr{P}$ is understood) for $\cup\{|\delta v||v \in \sum\}$. This is of course with reference to some **127**
centering η of $\mathscr{P}$. $\delta_{\mathscr{P}}(\sum)$ is a regular neighbourhood $|\mathscr{P}_{\sum}|$ in $|\mathscr{P}|$. If $\sum$ is a set consisting of single vertex x, we have the some what confusing

situation $\delta_{\mathscr{P}}(\{x\}) = |\delta\{x\}|$, where x denotes the 0-simplex with vertex x. In this case we will write $|\delta_{\mathscr{P}} x|$ or $|\delta x|$ for $\delta_{\mathscr{P}}(\{x\})$.

Let $\mathscr{N}$ be a subpresentation of a simplicial presentation and η be a centering of $\mathscr{P}$. Let $\mathscr{P}' = d(\mathscr{P},\eta)$ and $\mathscr{N}' = d(\mathscr{N},\eta)$ (still calling $\eta|\mathscr{N}$ as η). If $\sum$ is the set of vertices of $\mathscr{P}'$ consisting of the centers of elements of $\mathscr{N}$, then $\mathscr{P}'_{\sum} = \mathscr{N}' = d(\mathscr{N},\eta)$. $\mathscr{N}'$ is full in $\mathscr{P}'$. Given a centering of $\mathscr{P}' = d(\mathscr{P},\eta)$, we define

$$\mathscr{C}^* = |\delta(\eta C)|, \quad \text{for any} \quad C \in \mathscr{P}$$

and $\mathscr{N}^* = \delta_{\mathscr{P}'}(\sum) = \cup\{C^*|C \in \mathscr{N}\}$ is a regular neighbourhood of $|\mathscr{N}|$. We use the same notation ($\mathscr{N}^*$) even when $\mathscr{N}$ is *not* subpresentation, but a subset of $\mathscr{P}$. These are used in the last part of the chapter. As the particular centerings are not so important, we ignore them from the terminology whenever possible.

6.4 Collaring

To study regular neighbourhoods in more detail we need a few facts about collarings. This section is devoted to proving these.

Definition 6.4.1. Let A be a subpolyhedron of a polyhedron P. A is said
128 to be *collared* in P, if there is a polyhedral embedding h of $A \times [0,1]$ in P, such that

(i) $h(a,0) = \epsilon$ for all $a \in A$

(ii) the image of h is a neighbourhood of A in P. And the image of h is said to be a *collar* of A.

Definition 6.4.2. Let N be a subpolyhedron of a polyhedron P and let $B = Bd_P N$. N is said to be *bicollared* in P if and only if

(i) B is collared in N

(ii) B is collared in $\overline{P - N}$.

Definition 6.4.2′. Clearly this is equivalent to saying that there is a polyhedral embedding h of $B \times [-1,+1]$ in P such that

(i) $h(b,0) = b$, $b \in B$

(ii) $h(B \times (0,+1]) \subset P - N$

(iii) $h(B \times [-1,0]) \subset N$

(iv) the image of h is a neighbourhood of B in P.

Proposition 6.4.3. *If N is a regular neighbourhood of X in P, then N is bicollared in P.*

Proof. It is enough to prove this for some convenient regular neighbourhood of X. Let $\mathscr{P}$ be a simplicial presentation of P containing a full subpresentation $\mathscr{X}$ covering X and let $\varphi : P \to [0,1]$ be the usual map. We take N to be $\varphi^{-1}([0,\frac{1}{2}])$ clearly $Bd_P N = \varphi^{-1}(\frac{1}{2})$. Let us denote this by B. We can now show that $\varphi^{-1}([\frac{1}{4},\frac{3}{4}])$ is polyhedrally equivalent to $B \times [-1,+1]$ in the following way:

B has a regular presentation $\mathscr{B}$ consisting of all non empty sets $\sigma \cap$ 129
$\varphi^{-1}(\frac{1}{2})$ for $\sigma \in \mathscr{P}$.

Likewise $\varphi^{-1}([4,\frac{3}{4}])$ has a polyhedral presentation Q consisting of all non-empty sets of the following sorts:

$$\begin{aligned}
&\sigma \cap \varphi^{-1}(\frac{1}{4})\\
&\sigma \cap \varphi^{-1}(\frac{1}{4},\frac{1}{2})\\
&\sigma \cap \varphi^{-1}(\frac{1}{2})\\
&\sigma \cap \varphi^{-1}(\frac{1}{2},\frac{3}{4})\\
&\sigma \cap \varphi^{-1}(\frac{3}{4}) \quad \text{for} \quad \sigma \in \mathscr{P}
\end{aligned}$$

$\mathscr{J} = \{\{-1\},(-1,0),\{0\},(0,+1),\{+1\}\}$ is regular presentation of $[-1,+1]$. There is an obvious combinatorial isomorphism between Q and $\mathscr{B} \times \mathscr{J}$, which determines, is a appropriate centerings, a polyhedral equivalence between $B \times [-1,+1]$ and $\varphi^{-1}([\frac{1}{4},\frac{3}{4}]) \subset P$.

This shows that N is bicollared in P. □

Ex. 6.4.4. If A is collared in P, then any regular neighbourhood of A in P is a collar of A.

[Hint: Use 6.3.7 and 6.3.9].

Thus if N is a regular neighbourhood of X in P and $B = Bd_P N$, a regular neighbourhood of B in $\overline{P - N}$ is a collar of B.

Ex. 6.4.5. If N_1 is a regular neighbourhood of X in P and N_2 is a regular neighbourhood of N_1 in P, then $\overline{N_2 - N_1} = N_2 - \mathrm{Int}_P N$, is collar of $B_1 = Bd_P N_1$.

[Hint: Use 6.3.8 and 6.4.4].

130 **Ex. 6.4.6.** If N_1 is a regular neighbourhood of X in P, and N_2 is a regular neighbourhood of N_1 in P, then N_2 is a regular neighbourhood of X in P.

Ex. 6.4.7. (i) If N_1 and N_2 are two regular neighbourhoods of X in P with $N_1 \subset \mathrm{Int}_P N_2$, then $\overline{N_2 - N_1}$ is collar over $B_1 = Bd_P N_1$.

(ii) N_2 is a regular neighbourhood of N_1.

[Hint: Take two regular neighbourhoods N_2', N_1' of X, such that $N_2' - N_1'$ is a collar and try to push N_2 onto N_2' and N_1 onto N_1'].

The following remark will be useful later:

Ex. 6.4.8. Let N be bicollared in P and N' be a regular neighbourhood of N in P. Then there is an itopoty of P taking N onto N'. If $X \subset \mathrm{Int}_P N$, this isotopy can be chosen so as to fix X.

Definition 6.4.9. A pair (B, C) of polyhedra with $B \supset C$, is said to be a *cone pair* if there is a polyhedral equivalence of B onto a cone on C, which maps C onto C.

Clearly in such a case we can assume that the map on C is the identity. And if (B, C) is a cone pair, C is collared in B.

Definition 6.4.10. Let $A \subset P$ be polyhedra, and 'a' a point of A. Then a pair (L_P, L_A) is said to be a *link of a in* (P, A) if

(i) $L_A \subset L_P$

(ii) L_A is a link of a in A **131**

(iii) L_P is a link of a in P.

If (L'_P, L'_A) is another link of a in (P, A), then the standard mistake $L_P \to L'_P$ takes L_A onto L'_A, and therefore there is a polyhedral equivalence $L_P \to L_{P'}$ taking $L_A \to L_{A'}$. We shall briefly term this an equivalence of pairs $(L_P, L_A) \xrightarrow{\approx} (L_{P'}, L_{A'})$. So that, upto this equivalence, the link of a in (P, A) is unique.

Definition 6.4.11. Let $A \subset P$ be polyhedra. A is said to be *locally collared* in P if the link of a in (P, A) is a cone pair for every point $a \in A$.

Clearly $A \times 0$ is locally collared in $A \times [0, 1]$, and therefore if A is collared in P, it is locally collared. We will show presently that the converse is also true.

Definition 6.4.12. Let B be a subpolyhedron of $A \times [0, 1]$. B is said to be *cross section* if the projection $A \times [0, 1] \to A$, when restricted to B is $1 - 1$ and onto and so is a polyhedral equivalence $B \approx A$.

Proposition 6.4.13. *Let B be a cross-section of $A \times [0, 1]$ contained in $A \times (0, 1)$. Then there is a polyhedral equivalence $h : A \times [0, 1] \to A \times [0, 1]$, leaving $A \times 0$ and $A \times 1$ pointwise fixed, and taking B onto $A \times \frac{1}{2}$ and such that $h(a \times [0, 1]) = a \times [0, 1]$ for all $a \in A$.*

Remark . There is an obvious homeomorphism with these properties, but it is not polyhedral.

Proof. Let $p : A \times [0, 1] \to A$ be the first projection. Triangulate the **132**
polyhedral equivalence $p|B : B \to A$. Let $\mathscr{B}$ and $\mathfrak{a}$ be the simplicial presentations of B and A.

$$\mathscr{J} = \left\{\{0\}, \left(0, \frac{1}{2}\right), \left\{\frac{1}{2}\right\}, \left(\frac{1}{2}, 1\right), \{1\}\right\}$$

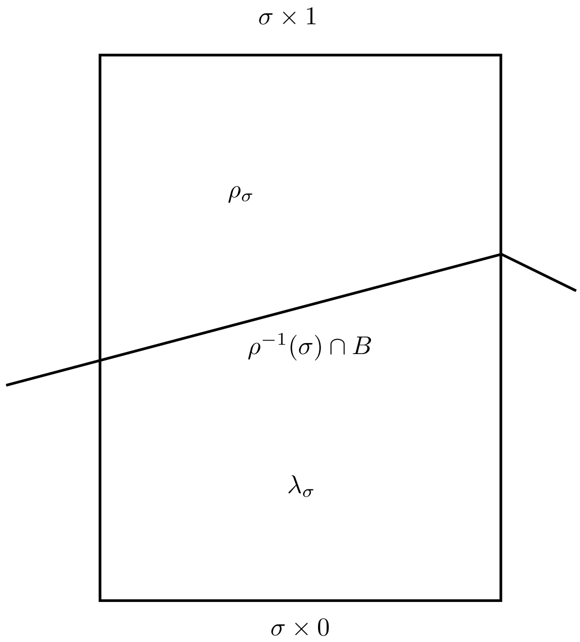
is a simplicial presentation of $[0, 1]$. Consider the centering η of $\mathfrak{a} \times \mathscr{J}$ given by $\eta(\sigma \times \tau) =$ (barycenter of σ, barycenter of τ), $\sigma \in \mathfrak{a}$, $\tau \in \mathscr{J}$.

We will define another regular presentation $\mathscr{C}$ of $A \times I$ as follows: For each $\sigma \in \mathfrak{a}$, $p^{-1}(\sigma)$ is the union of the following five cells:

$$\sigma \times 0, p^{-1}(\sigma) \cap B, \quad \sigma \times 1$$
$$\lambda_\sigma, \rho_\sigma;$$

where λ_σ is the region between $\sigma \times 0$ and $p^{-1}(\sigma) \cap B$ and ρ_σ is the region between $p^{-1}(\sigma) \cap B$ and $\sigma \times 1$. (Note that $p^{-1}(\sigma) \cap B \in \mathscr{B}$).

We take $\mathscr{C}$ to be the set of all these cells as σ varies over $\mathfrak{a}$. Choose a centering θ of $\mathscr{C}$, such that the first co-ordinate of each of the five cells above is the barycenter of σ.



Now there is an obvious combinatorial isomorphism $\mathscr{C} \approx \mathfrak{a} \times \mathscr{J}$; and if we choose the centerings described we obtain $h : A \times I \to A \times$
133 I which is simplicial relative to $d(\mathscr{C}, \theta)$ and $d(\mathfrak{a} \times \mathscr{J}, \eta)$, and has the desired properties. □

6.4.13 In this situation, define

$$\lambda_B = \{(a,t) \mid a \in A, t \in I, \exists b \in B, b = (a,s), t \leq s\}$$

i.e. this is all the stuff of the left of B. Then h takes λ_B onto $A \times [0, \frac{1}{2}]$, B onto $A \times \frac{1}{2}$. In particular B is collared in λ_B.

6.4.14 "Spindle Maps". Let $L \subset A$, with the cone on L and vertex 'a' contained in A. Call the cone S. Suppose $S - L$ is open in A (This is the case when a is a vertex of a simplicial presentation $\mathfrak{a}$ of A, and $L = |Lk(a, \mathfrak{a})|$ and $S = |St(a, \mathfrak{a})|$.

Let $\beta = I \to I$ be an imbedding with $\beta(1) = 1$. In this situation we define the "*spindle map*".

$$m(\beta, L, a) : A \times I \to A \times I$$

thus: on $L * [a \times I]$, it is the join of the identity map on L with the map $(a,t) \to (a, \beta(t))$ of $a \times I$. On the rest of $A \times I$ it is the identity map.

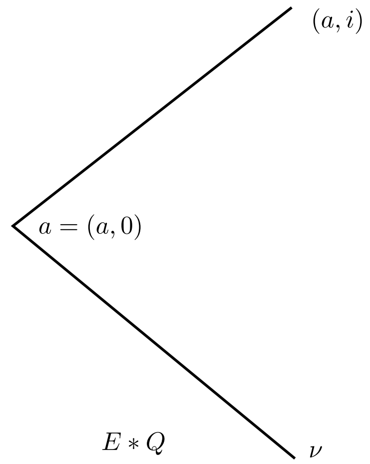
A spindle map m is an embedding, and commutes with the projection on A. If B is a cross section of $A \times I$ which does not intersect $A \times 1$, then $m(B)$ has these properties also.

Proposition 6.4.17. *Let $A \subset P$ be polyhedra. If A is locally collared in P, then A is collared in P.*

Proof. In $P \times [0,1]$, consider the subpolyhedron $Q = P \times 0 \cup A \times [0,1]$. We identify P with $P \times 0 \subset Q$. Let $\mathscr{P}$ be a simplicial presentation of P, in which a subpresentation $\mathfrak{a}$ covers A. **134**

Consider a vertex 'a' of $\mathfrak{a}$; let L_A and L_P denote $|Lk(a, \mathfrak{a})|$ and $|Lk(a, \mathscr{P})|$. Then (L_P, L_A) is a link of a in (P, A) and there is a polyhedral equivalence $\gamma : L_P \to L_A * v$ for some v, taking L_A onto L_A. We can make γ identity on L_A by composing with $(\lambda|L_A)^{-1} * \mathrm{id}_v$. And so we suppose γ/L_A is identity.

We can suppose that v is so situated (for example in a larger vector space) that $L_A * v$ and $L_A * (a,1)$ intersect only in L_A. Thus we have via γ and the identity on $L_A * (a,1)$, a polyhedral equivalence of $L_Q = L_P \cup L_A * (a,1)$ with $L_A * E$ where $E = \{v, (a,1)\}$, which is identity on



$L_A{*}(a, 1)$. Now $L_Q{*}a$ is a star of 'a' in Q, and via this p.e. is polyhedrally equivalent to $L_A * E * a$. We can find a polyhedral equivalence β of $E * a$ (which is equivalent to a closed 1-cell) leaving ν and $(a, 1)$ fixed and taking $(a, 0)$ to $(a, \frac{1}{2})$. Such a obviously takes $a \times [0, 1]$ onto $a \times [\frac{1}{2}, 1]$.

Take the join of β and the identity map L_A, this gives a polyhedral equivalence of $L_Q * a$ which is the identity on L_Q. Hence this can be extended to a polyhedral equivalence of Q by identity outside $L_Q * a$.
135 Let us call this equivalence of $Q, \beta_a \cdot \beta_a(A \times I) \subset A \times I$, and $\beta_a|A \times I$ is a spindle map.

Now take the composition h in any order of all such β_a, with 'a' running over all the vertices of $\mathfrak{a}$. This maps $A = A \times 0$ into a cross section $h(A) = B$ of $A \times [0, 1]$ which does not intersect $A \times 1$ or $A \times 0$. Finally $h(P) \cap A \times I = \lambda B$.

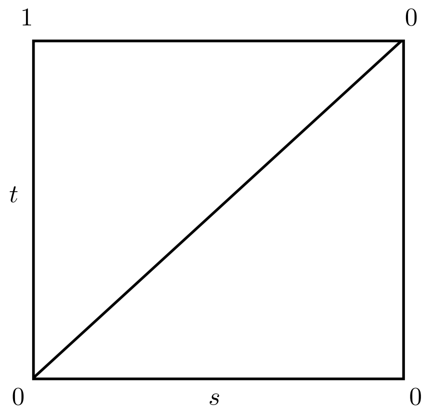
B is collared in λB, and so in $h(P)$. Then, taking h^{-1} we see that A is collared in P. ◻

Corollary 6.4.16. *If M is a P.L. Manifold with boundary M, then ∂M is collared in M.*

Now, an application of the corollary:

Proposition 6.4.17. *If h is an isotopy of ∂M, then h extends to an isotopy H of M.*

Proof. Let $\beta : I \times I \to I$ be the map given by $\beta(s,t) = \mathrm{Max}(t-s, 0)$. This is polyhedral, e.g. the diagram shows that triangulations and the images of the vertices.



$\beta(s,0) = 0$, $\beta(1,t) = 0$, $\beta(0,t) = t$. Define $H = (\partial M \times I) \times I \to (\partial M \times I) \times I$ by $H((x,s),t) = ((h_{\beta(s,t)}(x), s), t)$. This is polyhedral.

$$H((x,s),0) = ((h_{\beta(s,0)}(x), s), 0) = ((x,s),0),$$

since $h_0 = \mathrm{Id}$. Hence $H_0 = \mathrm{Id}$ of $\partial M \times I$.

$$H((x,0),t) = ((h_{\beta(0,t)}(x), 0), t) = ((h_t(x), 0), t).$$

Thus H extends the isotopy $\partial M \times 0$ given by h (identifying ∂M and $\partial M \times 0$). And 136

$$\begin{aligned} H((x,1),t) &= ((h_{\beta(1,t)}(x), 1), t) \\ &= ((x,1),t) \quad \text{since} \quad \beta(1,t) = 0. \end{aligned}$$

Hence $H|\partial M \times 1$ is identity. Hence the isotopy h of ∂M extends to an isotopy H of any collar so that at the upper end of the collar it is identity again, and therefore it can be extended inside. Thus h extends to an isotopy of M. □

6.5 Absolute Regular Neighbourhoods and some Newmanish Theorems

Definition 6.5.1. A pair of a polyhedra (P, A) is said to be an *absolute regular neighbourhood* of a polyhedron X, if

(i) $X \subset P - A$

(ii) $P \times 0$ is a regular neighbourhood of $X \times 0$ in $P \times 0 \cup A \times [0, 1] \subset P \times [0, 1]$.

Hence A is collared in P.

Probably, it will be more natural to consider X, P and A in an ambient polyhedron M in which P is a neighbourhood of X as in links and stars. But, after the definition of regular neighbourhood, absolute regular neighbourhood is just a convenient name to use in some tricky situations.

Ex. 6.5.2. If (P, A) is an absolute regular neighbourhood of X and if $h : P \to P'$ is a polyhedral equivalence, then $(P', h\ A)$ is an absolute regular neighbourhood of $h\ X$.

Ex. 6.5.3. If N is a regular neighbourhood of X in P, and $B = Bd_P N$, then (N, B) is an absolute regular neighbourhood of X.

137 **Ex. 6.5.4.** Let $P \subset Q$, and suppose that (P, A) is an absolute regular neighbourhood of X, and $P - A$ is open in Q, and A is locally collared in $Q - (P - A)$. Then P is regular neighbourhood of X in Q.

Ex. 6.5.5. Let $C(A)$ be the cone on A with vertex v. Then $(C(A), A)$ is an absolute regular neighbourhood of v.

In particular if D is an n-cell, $(D, \partial D)$ is an absolute regular neighbourhood of any point $x \in D - \partial D$.

Theorem 6.5.6. *If D is an n-cell, M a PL-manifold, $D \subset \operatorname{Int} M$, then D is a regular neighbourhood of any $x \in D - \partial D$ in M.*

Corollary 6.5.7. *If D is an n-cell in an n-sphere S, then $\overline{S - D}$ is an n-cell.*

Proof of the theorem: The proof of the theorem is by induction on the dimension of M; we assume the theorem as well as the corollary for $n - 1$.

(i) First we must show that $D - \partial D$ is open in M. If we look at the links, this would follow if we know that a polyhedral imbedding of an $(n-1)$-sphe re in an $(n-1)$-sphere is necessarily onto. And this can be easily seen by looking at the links again and induction. (see 4.4 in particular 4.4.14 and 4.4.17(a)).

(ii) If we know that ∂D is collared in $M - \operatorname{int} D$ (it is collared in D), we are through by 6.5.4. For this, it is enough to show that ∂D is locally collared in $M - \operatorname{int} D$. Consider a link of a in M, say S^{n-1},
such that a link of 'a' in D is an $(n-1)$-cell $D^{n-1}_a \subset S^{n-1}_a$, with **138**
$D^{n-1}_a \cap \partial D = \partial D^{n-1}_a$. It is clearly possible to choose such links (see 4.4.17(b)). Now, a link of 'a' in $M - \operatorname{int} D$ is $S^{n-1}_a - (D^{n-1}_a - \partial D^{n-1}_a)$. As in (i) $D^{n-1}_a - \partial D^{n-1}_a$ is open in S^{n-1}_a and therefore the link of a in $M - \operatorname{int} D$ is $\overline{S^{n-1}_a - D^{n-1}_a}$. But by the corollary to the theorem in the $(n-1)$-case, this is an $(n-1)$-cell, say Δ^{n-1} and it meets D in $\partial D^{n-1}_a = \partial\Delta^{n-1}$. And $(\Delta^{n-1}, \partial\Delta^{n-1})$ is equivalent to $(C(\partial\Delta^{n-1}), \partial\Delta^{n-1})$. Therefore ∂D is locally collared in $M - \operatorname{int}$ D and we are through.

Proof of the corollary assuming the theorem: Represent S^n, a standard n-sphere as a suspension of S^{n-1}, a standard $(n-1)$-sphere, and observe that the lower hemisphere (say D_s) is a regular neighbourhood of the south pole, say s. Let f be a polyhedral equivalence of S to S^n taking a point $x \in D - \partial D$ to the south pole s. By the theorem D is a regular neighbourhood of x, therefore $f(D)$ is a regular neighbourhood of the s in S^n. By 6.3.2 there is a polyhedral equivalence p of S^n such that $p(D_s) = f(D)$. Therefore $f(\overline{S - D}) = \overline{f(S) - f(D)} = \overline{S^n - p(D_s)} = \overline{p(S^n) - p(D_s)} = \overline{p(S^n - D_s)} = p(D_n)$, where D_n denotes the upper hemisphere. Therefore $p^{-1} \cdot f(\overline{S - D}) = D_n$ or $\overline{S - D}$ is a n-cell.

Ex. 6.5.8. Corollary. If M is a PL n-manifold and D_1, D_2 are two n-cells contained in the interior of the same component of M, then there is an isotopy h of the identity map of M, such that $h(D_1) = D_2$.

139 We usually express this by saying that "any two n-cells in the interior of the same component of M are equivalent" or that they are "equivalent by an isotopy of M".

If M is a PL n-manifold, ∂M its boundary, then by 6.5.8, any two $(n-1)$-cells in the same component of ∂M are equivalent by an isotopy of ∂M. Since this is actually an isotopy of the identity, by 6.4.17 we can extend it to M. Thus

Proposition 6.5.9. *Any two $(n-1)$-cells in the same component of ∂M are equivalent by an isotopy of M.*

This immediately gives

Ex. 6.5.10. If D is an n-cell and Δ an $(n-1)$-cell in ∂D, then (D, Δ) is a cone pair (That is, there is a polyhedral equivalence of (D, Δ) and $(C(\Delta), \Delta)$. And we have seen such a polyhedral equivalence can be assumed to be identity on Δ).

This can also be formulated as:

Ex. 6.5.10[1] If Δ_i is an $(n-1)$-cell in the boundary of D_i, an n-cell, $i = 1, 2$, any polyhedral equivalence $\Delta_1 \to \Delta_2$ can be extended to a polyhedral equivalence $D_1 \to D_2$.

Also from 6.5.9, it is easy to deduce if Δ is any $(n-1)$-cell in ∂M, then there is at least one n-cell D in M such that $D \cap \partial M = \Delta \subset \partial D$. From this follows the useful proposition:

Ex. 6.5.11. If M is a PL n-manifold and D an n-cell with $M \cap D = \partial M \cap \partial D =$ an $(n-1)$-cell, then $M \cup D$ is polyhedrally equivalent to M. Moreover, the polyhedral equivalence can be chosen to be identity outside any given neighbourhood of $M \cup D$ in M.

140 The methods of the proof of the theorem 6.5.6, can be used to prove the following two propositions, which somewhat clarify the nature of regular neighbourhoods in manifolds:

Ex. 6.5.12. Let M be a PL-manifold, ∂M its boundary (possibly $\emptyset$), and N a regular neighbourhood of X in M. Then

(a) N is a PL-manifold with (non-empty) boundary unless X is a union of components of M.

(b) If $X \subset M - \partial M$, then $N \subset M - \partial M$, the interior of M.

(c) If $X \cap \partial M \neq \emptyset$, $N \cap \partial M$ is a regular neighbourhood of $X \cap \partial M$ in ∂M.

(d) In case (c), $Bd_M N$ is an $(n-1)$-manifold, meeting ∂M in an $(n-2)$-manifold $\partial N'$, where $N' = N \cap \partial M$.

[Note that $\mathrm{int}_M N$ and $bd_M N$ denote the interior and boundary of N in the topology of M. On the otherhand if N is a PL-manifold int N and ∂N denotes the sets of points of N whose links are spheres and cells respectively].

Hint: Use 4.4.8.

Ex. 6.5.13. If N is a regular neighbourhood of X in M, a PL-manifold with $X \subset \mathrm{int}\, M$, and N' is polyhedrally equivalent to N and located in the interior of a PL-manifold M_2 of the same dimension as M, then N' is a regular neighbourhood of X' in M_2, where X' is the image of X under the polyhedral equivalence $N \to N'$.

Ex. 6.5.14. A is any polyhedron, and I the standard 1-cell $(A \times I, A \times 1)$ **141**
is an absolute regular neighbourhood of $A \times 0$. If $0 < \mathcal{L} < 1$, then $(A \times I, A \times \{0, 1\})$ is an absolute regular neighbourhood of $A \times \mathcal{L}$.

Ex. 6.5.15. The union of two n-manifolds intersecting in an $(n-1)$ submanifold of their boundaries is an n-manifold.

6.6 Collapsing

Definition 6.6.1. Let $\mathscr{P}$ be a regular presentation. *A free edge* of $\mathscr{P}$ is some $E \in \mathscr{P}$ such that there exists one and only one $A \in \mathscr{P}$ with $E < A$. We may term A the *attaching membrane* of the free edge E. It is clear that A is not in the boundary of any other element of $\mathscr{P}$; for if $A < B$, then $E < B$. It is easily proved that $\dim A = 1 + \dim E$.

The set $\mathscr{P} - \{E, A\}$ is again a regular presentation, and is said to be obtained from $\mathscr{P}$ by an *elementary collapse* at the free edge E.

Definition 6.6.2. We say that a polyhedral presentation $\mathscr{P}$ *collapses* (*combinatorially*) *to* a polyhedral presentation Q, and write $\mathscr{P} \searrow Q$, if there exists a finite sequence of presentations

$$\mathscr{P}_1, \ldots, \mathscr{P}_k \quad \text{with} \quad \mathscr{P} = \mathscr{P}_1 \quad \text{and} \quad \mathscr{P}_k = Q$$

and cells $E_1, \ldots, E_{k-1}$, $E_i \in \mathscr{P}_i$ s.t. $\mathscr{P}_i$ is obtained from $\mathscr{P}_{i-1}$ be an elementary collapse at E_{i-1}.

Proposition 6.6.3. *If Q is obtained from $\mathscr{P}$ by an elementary collapse at*
142 *E; and if $\mathscr{P}'$ is obtained from $\mathscr{P}$ by bisecting a cell C by a bisection of space $(L; H+, H-)$ and if $Q' \subset \mathscr{P}'$ is the subpresentation with $|Q'| = |Q|$, then $\mathscr{P}' \searrow Q'$. [Remark: Recall that, we have been always dealing with Euclidean polyhedra].*

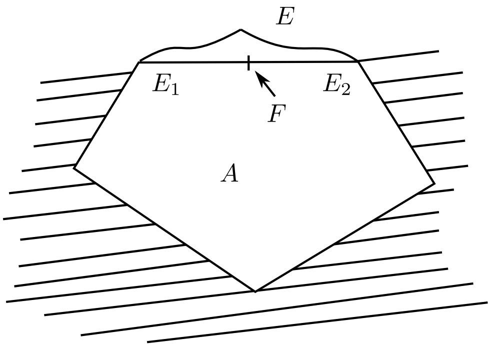
Proof. If the bisection is trivial there is nothing to prove, so suppose that the bisection is non trivial. Then there are three cases.

Case (i) C is neither E nor A. In this case, E is a free edge of $\mathscr{P}'$ with attaching membrane A, and $Q' = \mathscr{P}' - \{E, A\}$; thus Q' is obtained from $\mathscr{P}'$ by an elementary collapse.

Case (ii) $C = E$. Define $E_1 = H + \cap E$, $E_2 = H - \cap E$, $F = L \cap E$. Then we have

$$\begin{tikzcd}[column sep=small, row sep=small] & E_1 \arrow[dl] & \\ F & & A \arrow[ul] \arrow[dl] \\ & E_2 \arrow[ul] & \end{tikzcd}$$

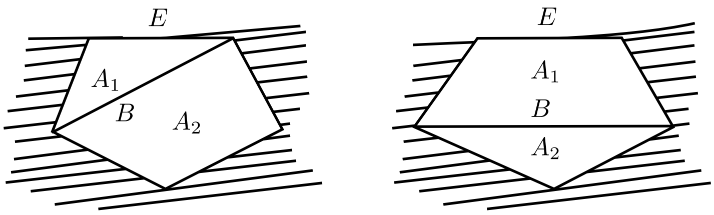
and no other cells of $\mathscr{P}'$ are greater than F, E_1, E_2 or A.



Thus E_1 is a free edge of $\mathscr{P}'$ with attaching membrane A; F is a free edge of $\mathscr{P}' - \{E_1, A\}$ with attaching membrane E_2. The result of these two elementary collapses is $Q = Q'$.

Case (iii) $C = A$ **143**

Define $A_1 = H + \cap A$, $A_2 = H - \cap A$, $B = L \cap A$. Now $\partial A_1 \cup \partial A_2$ contains ∂A, and therefore either ∂A_1 or ∂A_2 intersects E; say, $\partial A_1 \cap E \neq \emptyset$. Then $\mathscr{P}'$ being regular, we must have $E < A_1$; then for dimensional reasons, $\dim E = \dim B$, we cannot have $E < B$ hence $E \subset H+$; and so it is impossible to have $E < A_2$. In summary, $E < A_1 > B < A_2$.



Thus E is a free face of $\mathscr{P}'$ with attaching membrane A_1; B is a free face of $\mathscr{P}' - \{E, A_1\}$ with attaching membrane A_2. The result of these two elementary collapses is Q'. □

Proposition 6.6.4. *If $\mathscr{P} \searrow Q$, and $\mathscr{P}'$ is obtained from by a finite sequence of bisections of cells, and Q' is the subpresentation of $\mathscr{P}'$ defined by $|Q'| = |Q|$; then $\mathscr{P}' \searrow Q'$.*

Proof. The proof is by induction, first, on the number of collapses in $\mathscr{P} \searrow \mathcal{Q}$, and second, on the number of bisections involved. The inductive step is 6.6.3. $\square$

Definition 6.6.5. We say that a polyhedron P *collapses (geometrically) to a subpolyhedron* Q, if there is a regular presentation $\mathscr{P}$ of P with a subpresentation $\mathcal{Q}$ covering Q, such that $\mathscr{P}$ collapses combinatorially to $\mathcal{Q}$. We write $P \searrow Q$.

144 This notion is polyhedrally invariant:

Proposition 6.6.6. *If $P \searrow Q$, and $\alpha : P \to X$ is a polyhedral equivalence, then $X \searrow \alpha(Q)$.*

$\mathcal{L}$ α

Proof. There are regular presentations $\mathscr{P}$, $\mathcal{Q}$ of P and Q, with $\mathscr{P} \searrow \mathcal{Q}$ combinatorially, and simplicial presentations $\mathscr{S}$, $\mathscr{X}$ of P and X with $\mathcal{L}$ simplicial relative to $\mathscr{S}$ and $\mathscr{X}$. There is a regular presentation $\mathscr{P}'$ of P refining $\mathscr{P}$ and $\mathscr{S}$, and obtained from $\mathscr{P}$ (also from $\mathscr{S}$ but we do not need it in this proposition) by a finite sequence of bisections. Hence if $\mathcal{Q}'$ is the subpresentation of $\mathscr{P}'$ covering Q, then $\mathscr{P}' \searrow \mathcal{Q}'$, by 6.6.4. Since $\mathcal{L}$ is one-to-one and linear on each element of $\mathscr{P}'$, the set $\mathcal{L}(\mathscr{P}') = \{\mathcal{L}(C) \mid C \in \mathscr{P}'\}$ is a regular presentation of X, which is combinatorially isomorphic to $\mathscr{P}'$; and $\mathcal{L}(\mathcal{Q}')$ is subpresentation covering $\mathcal{L}(Q)$, which is combinatorially isomorphic to $\mathcal{Q}'$. Threrefore $\mathcal{L}(\mathscr{P}) \searrow \mathcal{L}(\mathcal{Q}')$ or $X \searrow (Q)$. $\square$

Proposition 6.6.7. *If $P_1 \searrow P_2$, and $P_2 \searrow P_3$, then $P_1 \searrow P_2$.*

Proof. Let $\mathscr{P}_1$, $\mathscr{P}_2$ be presentation of P_1, P_2 with $\mathscr{P}_1 \searrow \mathscr{P}_2$, and $\mathscr{P}_3$, $\mathscr{P}_4$ be presentations of P_2, P_3 with $\mathscr{P}_3 \searrow \mathscr{P}_4$. By 1.10.6 there is a regular refinement $\mathcal{Q}$ of $\mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{P}_3 \cup \mathscr{P}_4$, and subpresentations $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$, $\mathcal{Q}_4$ of with $|\mathscr{P}_i| = |\mathcal{Q}_i|$, $\mathcal{Q}_i$ obtained from $\mathscr{P}_i$ by a sequence of bisections. Clearly $\mathcal{Q}_2 = \mathcal{Q}_3$ and by 6.6.4, $\mathcal{Q}_1 \searrow \mathcal{Q}_2$, and $\mathcal{Q}_3 \searrow \mathcal{Q}_4$ and therefore $P_1 \searrow P_2$. $\square$

Proposition 6.6.8. *If N is a regular neighbourhood of X in P, then $N \searrow X$.*

Proof. By virtue of 6.6.6 and the definition of regular neighbourhood, it is enough to look at any particular N. Let $\mathscr{P}$ be a simplicial presentation of P with a full subpresentation $\mathscr{X}$ covering X; and let $\varphi : P \to [0,1]$ be the usual map. Take $N = \varphi^{-1}([0, \frac{1}{2}])$. 145

Let $\sum$ denote all the simplexes of $\mathscr{P}$ having vertices both in $\mathscr{X}$ and $(\mathscr{P} - \mathscr{X})$. We prove $N \searrow X$ by induction on the number of elements of $\sum$. If $\sum = \emptyset$, then $N = X$, and there is nothing to do. Hence we can start the induction.

Let

$$\mathscr{N} = \mathscr{X} \cup \{\sigma \cap \varphi^{-1}((0, \frac{1}{2})) \mid \sigma \in \sum\}$$
$$\cup \{\sigma \cap \varphi^{-1}(\frac{1}{2}) \mid \sigma \in \sum\}$$

Then $\mathscr{N}$ is a regular presentation of N. If σ is an element of of maximal dimension, $\sigma \cap \varphi^{-1}(\frac{1}{2})$ is a free edge of $\mathscr{N}$ with attaching membrane $\sigma \cap \varphi^{-1}((0, \frac{1}{2}))$. (Note that σ is a *principal* simplex of $\mathscr{P}$ i.e. not the face of any other simplex). After doing the elementary collapse we are left with $\mathscr{N}'$. Now $\mathscr{P} - \{\sigma\} = \mathscr{P}'$ is a regular presentation containing $\mathscr{X}$, and the corresponding $\sum' = \sum - \{\sigma\}$. Hence inductively $\mathscr{N}' \searrow \mathscr{X}$. And so, $\mathscr{N} \searrow \mathscr{X}$. □

Ex. 6.6.9. Let N' be a neighbourhood of X in P, (all polyhedra). If $N' \searrow X$, then there is a regular neighbourhood N of X in P, $N \subset \operatorname{Int}_P N'$ such that $N' \searrow N$.

6.7 Homogeneous Collapsing

Let $\mathscr{P}$ be a regular presentation, with $E, A \in \mathscr{P}$, $E < A$ and $\dim A = \dim E + 1$.

Recall the definition of $\lambda_{\mathscr{P}} E$. This is defined, relative to some centering η of $\mathscr{P}$, to be the full subpresentation of $d\mathscr{P}$ whose vertices are, $\{\eta\mathscr{C} \mid E < C \in \mathscr{P}\}$. 145

Definition 6.7.1. Let E, A, $\mathscr{P}$; be as above and η be a centering of $\mathscr{P}$. (E, A) is said to be *homogenous* in $\mathscr{P}$, if there is a polyhedron X and a polyhedral equivalence $f : |\lambda_{\mathscr{P}} E| \to X * \{u, w\}$ a suspension of X, such that $f(\eta A) = w$.

It is easily seen that if this is true for one centering of $\mathscr{P}$, then it is true for any other centering of $\mathscr{P}$; hence "(E, A) is homogeneous in $\mathscr{P}$" is well defined.

Definition 6.7.2. Let $\mathscr{X} \subset \mathscr{N}$ be subpresentations of $\mathscr{P}$. We say that $\mathscr{N}$ *collapses to* $\mathscr{X}$ *homogeneously (combinatorially) in* $\mathscr{P}$, if there is a finite sequence of subpresentations of $\mathscr{P}$,

$$\mathscr{N}_1, \ldots, \mathscr{N}_k$$

and pairs of cells $(E_1, A_1), \ldots, (E_{k-1}, A_{k-1})$, E_i, $A_i \leftarrow \mathscr{N}_i$ such that

(1) $\mathscr{N}_1 = \mathscr{N}$, $\mathscr{N}_k = \mathscr{X}$

(2) $\mathscr{N}_{i+1}$ is obtained from $\mathscr{N}_i$ by an elementary collapse at E_i, a free edge of $\mathscr{N}_i$ with attaching membrane A_i, for $i = 1, \ldots, k-1$ and

(3) (E_i, A_i) is homogeneous in $\mathscr{P}$, for $i = 1, \ldots, k-1$.

Proposition 6.7.3. *If $\mathscr{P}'$ is obtained from $\mathscr{P}$ by bisecting a cell C by a bisection of space $(L; H+, H-)$: and if $\mathscr{X} \subset \mathscr{N} \subset \mathscr{P}$, with $\mathscr{X}$ obtained from $\mathscr{N}$ by an elementary collapse at a free edge E with attaching membrane A, where (E, A) is homogeneous in $\mathscr{P}$; and if $\mathscr{N}'$, $\mathscr{X}'$*
146 *are the subpresentations of $\mathscr{P}'$ covering $|\mathscr{N}|$ and $|\mathscr{X}|$; then $\mathscr{N}' \searrow \mathscr{X}'$ homogeneously in $\mathscr{P}'$.*

Proof. If the bisetion in trivial there is nothing to prove. If it is not trivial, there are three cases as in the proof of proposition 6.6.3.

Case 1: C is neither E nor A. In this case the only problem is to show that (E, A) is homogeneous in $\mathscr{P}'$. Let us suppose that everything is occuring in a vector space V of $\dim n$; and let $\dim E = k$. Then there is an orthogonal linear manifold M of dimension $(n-k)$, intersecting E in a single point $\eta(E) = e$, say. It is fairly easy to verify that in such a situation if $E < D$, then $D \cap M \neq \emptyset$.

If we now choose centerings of $\mathscr{P}$ and $\mathscr{P}'$ so that whenever $D \cap M \neq \emptyset$, we have the center of D belonging to M, then defining

$$Q = \{D \cap M \mid D \cap M \neq \emptyset, D \in \mathscr{P}\}$$

and Q' similarly with respect to $\mathscr{P}'$, we will have:

$$\lambda_{\mathscr{P}}(E) = \lambda_Q(e)$$
$$\lambda_{\mathscr{P}'}(E) = \lambda_{Q'}(e).$$

and Q, Q' are regular presentations of $|\mathscr{P}| \cap M$. Hence both $|\lambda_{\mathscr{P}}(E)|$ and $|\lambda_{\mathscr{P}'}(E)|$ are links of e in $|\mathscr{P}| \cap M$, and hence polyhedrally equivalent (by an approximation to the standard mistake); if we choose a center of A the same in both case, we get a polyhedral equivalence taking ηA to ηA. Finally, by hypothesis $|\lambda_{\mathscr{P}}(E)|$ is equivalent to a suspension with ηA as a pole; and so $|\lambda \mathscr{P}'(E)|$ has the same property, and (E, A) is homogeneous in $\mathscr{P}'$.

Case 2: $C = E$; we define E_1, E_2, F as in the proof of 6.6.3. We have to show that (E_1, A) and (F, E_2) are homogeneous in $\mathscr{P}'$. **147**

That (E_1, A) is homogeneous in $\mathscr{P}'$ follows from the fact $|\lambda_{\mathscr{P}}(E)| = |\lambda_{\mathscr{P}'}(E_1)|$ (with appropriate centerings) because any $D > E_1$ in $\mathscr{P}'$ is an element of $\mathscr{P}$ which is $> E_1$ and hence, $\mathscr{P}$ being regular $> E$.

That (F, E_2) is homogeneous in $\mathscr{P}'$, we see by the formula:

$$\lambda_{\mathscr{P}'}(F) = \lambda_{\mathscr{P}}(E) * \{\eta E_1, \eta E_2\}$$

(calling the appropriate centering of $\mathscr{P}'$ also η).

Case 3: $C = A$; we define A_1, A_2, B as in the proof of 6.6.3. We have to show that (E, A_1) and (B, A_2) are homogenous.

There is a simplicial isomorphism $\lambda_P(E) \approx \lambda_{\mathscr{P}'}(E)$ taking $\eta(A)$ onto $\eta(A_1)$. And as (E, A) is homogeneous in $\mathscr{P}$, we have (E, A_1) is homogeneous in $\mathscr{P}'$.

That (B, A_2) is homogeneous in $\mathscr{P}'$ we see by a formula like that in case 2:

$$\lambda_{\mathscr{P}'}(B) = \lambda_{\mathscr{P}}(A) * \{\eta A_1, \eta A_2\}.$$

□

Proposition 6.7.4. *If $\mathscr{N} \searrow \mathscr{X}$ homogeneously in $\mathscr{P}$, and $\mathscr{P}'$ is obtained from $\mathscr{P}$ by a finite sequence of bisections of space, and $\mathscr{N}'$, $\mathscr{X}'$ are the subpresentations of $\mathscr{P}'$ covering $|\mathscr{N}|$ and $|\mathscr{X}|$, then $\mathscr{N}' \searrow \mathscr{X}'$ homogeneously in $\mathscr{P}'$.*

This follows from 6.6.3, as 6.6.4 from 6.6.3.

148 **Definition 6.7.5.** Let P be a polyhedron, and X, N subpolyhedra of P. N is said to *collapse homogeneously (geometrically) to X in P*, if there are regular presentation $\mathscr{X} \subset \mathscr{N} \subset \mathscr{P}$ covering X, N and P respectively such that $\mathscr{N}$ collapses homogeneously to $\mathscr{X}$ combinatorially in $\mathscr{P}$.

We write $N \searrow X$ *homogeneously* in P. This definition is again polyhedrally invariant:

Proposition 6.7.6. *If $N \searrow X$ homogeneously in P, and $\mathcal{L} : P \to Q$ is a polyhedral equivalence, then $\mathcal{L}(N) \to \mathcal{L}(X)$ homogeneously in Q.*

This follows from 6.7.4 as 6.6.6 from 6.6.4.

Proposition 6.7.7. *If N is a regular neighbourhood of X in P, then $N \searrow X$ homogeneously in P.*

Proof. As in 6.6.8, we start with a simplicial presentation $\mathscr{P}$ of P in which a full subpresentation $\mathscr{X}$, covers X, and take $N = \varphi^{-1}([0, \frac{1}{2}])$ where $\varphi : P \to [0, 1]$ is the usual map. By virtue of 6.7.6, and the definition of regular neighbourhood, it is enough to prove that this $N \searrow X$ homogeneously.

Let $\mathscr{N}$ be the regular presentation of N consisting of cells of the form:

$$\text{simplexes of } \mathscr{X}$$
$$\sigma \cap \varphi^{-1}((0, \frac{1}{2})), \quad \text{for} \quad \sigma \in \mathscr{P} \quad \text{with} \quad \varphi(\sigma) = (0, 1)$$
$$\sigma \cap \varphi^{-1}(\frac{1}{2}), \quad \text{for} \quad \sigma \in \mathscr{P} \quad \text{with} \quad \varphi(\sigma) = (0, 1)$$

Define $\mathscr{P}'$ to consist of

all simplexes of $\mathscr{X}$,
all simplexes of $\mathscr{P}$ which have no vertices in $\mathscr{X}$.

$$\left.\begin{array}{l}\sigma \cap \varphi^{-1}((0, \frac{1}{2})) \\ \sigma \cap \varphi^{-1}((\frac{1}{2})) \\ \sigma \cap \varphi^{-1}(\frac{1}{2}, 1)\end{array}\right\} \text{ for } \sigma \in \mathscr{P} \quad \text{with} \quad \varphi(\sigma) = (0, 1)$$

$\mathscr{P}'$ is a regular presentation of P which refines $\mathscr{P}$, and has as subpre- 149
sentations $\mathscr{N}$ and $\mathscr{X}$. $\mathscr{N}$ and $\mathscr{X}$ are the same as in proposition 6.6.8, and therefore we know that $\mathscr{N} \searrow \mathscr{X}$. Now, the claim is $\mathscr{N} \searrow \mathscr{X}$ homogeneously in $\mathscr{P}'$. In otherwords, if $E = \sigma \cap \varphi^{-1}(\frac{1}{2})$, $A = \sigma \cap \varphi^{-1}((0, \frac{1}{2}))$, where $\sigma \in \mathscr{P}$ with $\varphi(\sigma) = (0, 1)$, we have to show that (E, A) is homogeneous in $\mathscr{P}'$. In fact denoting by $\mathscr{B}$ the subpresentation of $\mathscr{P}'$ covering $\varphi^{-1}(\frac{1}{2})$, we have

$$\lambda_{\mathscr{P}'}(E) = \lambda_{\mathscr{B}}(E) * \{\eta A, \eta A'\}, \quad \text{where}$$
$$A' = \sigma \cap \varphi^{-1}((\frac{1}{2}, 1)).$$

□

6.8 The Regular Neighbourhood Theorem

We have seen that if N is a regular neighbourhood of X in P, then

(1) $X \subset \operatorname{int}_P N$

(2) N is bicollared in P

(3) $N \searrow X$ homogeneously in P.

Conversely.

6.8.1 The Regular Neighbourhood Theorem

If X N P are polyhedra such that

(1) $X \subset \operatorname{int}_P N$

(2) N is bicollared in P

(3) $N \searrow X$ homogeneously in P **150**

then N is a regular neighbourhood of X in P.

The proof will start with some technicalities which exploit the homogeneity of the collapsing (The X's, P's etc. occuring mean-while should not be confused with the X, P of the theorem).

Proposition 6.8.2. *Let $Y \subset X$ be polyhedra, and let $P = X * \{v, w\}$ a suspension of X. Then a regular neighbourhood of $Y * v$ in P is a regular neighbourhood of v in P. [In other words, a regular neighbourhood of a subcone of a suspension is a regular neighbourhood of one of the poles].*

Proof. Let $C_1(X)$ denote $X * v$ and let $\varphi : C_1(X) \to [0, 1]$ be the join of the maps $X \to 1$ and $v \to 0$. For any $Z \subset X$, $C_{\mathcal{L}}(Z)$ for $0 < \mathcal{L} < 1$ will denote the set of points $\{(1 - t)v + tz \mid z \in Z, 0 \leq t \leq \mathcal{L}\}$. If Z is a subpolyhedron $C_{\mathcal{L}}(Z) = (Z * v) \cap \varphi^{-1}([0, \mathcal{L}])$.

By 6.3.7, it is enough to prove the proposition for some regular neighbourhood of $X * v$. Hence, by a couple of maps, it is enough to show that $C_{5/8}(X)$ is a regular neighbourhood of $C_{\frac{1}{2}}(Y)$ in $C_1(X)$. (It is clearly a regular neighbourhood of v in $C_1(X)$).

Let $\mathscr{X}$ be a simplicial presentation of X, containing a subpresentation $\mathscr{Y}$ covering Y. We define a regular presentation of $C_1(X)$ to consist of:

$$\begin{array}{ll} \{v\}_\sigma & \text{for} \quad \sigma \in \mathscr{X} \\ \sigma\{v\} & \text{for} \quad \sigma \in \mathscr{X} - \mathscr{Y} \\ \left.\begin{array}{l} \sigma\{v\} \cap \varphi^{-1}((0, \frac{1}{2})) \\ \sigma\{v\} \cap \varphi^{-1}(\frac{1}{2}) \\ \sigma\{v\} \cap \varphi^{-1}(\frac{1}{2}, 1) \end{array}\right\} & \text{for} \quad \sigma \in \mathscr{Y} \end{array}$$

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Then $\mathscr{P}$ has a subpresentation Q covering $\subset \frac{1}{2}(Y)$, and for each $A \in \mathscr{P} - Q$ with $\overline{A} \cap C_{\frac{1}{2}}(Y) \neq \emptyset$, $\varphi(A)$ includes the interval $(\frac{1}{2}, 1)$. Choose a centering η of $\mathscr{P}$, so that for all $A \in \mathscr{P} - Q$ with $\overline{A} \cap C_{\frac{1}{2}}(Y) \neq \emptyset$, $\varphi(\eta A) = \frac{3}{4}$.

Then $d(\mathscr{P},\eta)$ has the property that if τ is a simplex with vertices both in dQ and in $d\mathscr{P} - dQ$, then $\varphi(\tau)$ contains $(\frac{1}{2},\frac{3}{4})$, and dQ is full in $d\mathscr{P}$. Choose a centering θ of $d\mathscr{P}$ so that for $\tau \in d\mathscr{P}$ with vertices in and out of ∂Q, $\varphi(\theta\tau) = \frac{5}{8}$. Now $N = |N_{d\mathscr{P}}(dQ,\theta)| = \varphi^{-1}([0,5/8])$; and thus $\varphi^{-1}([0,5/8])$ is a regular neighbourhood of both $C_{\frac{1}{2}}(Y)$ and v. ◻

Now, let P be a polyhedron and $\mathscr{P}$ a simplicial presentation of P. Let $\sum$ be any set of vertices of $\mathscr{P}$ and η a centering of $\mathscr{P}$. Recall the definition of $\delta_{\mathscr{P}}(\sum)$ and $\mathscr{P}_{\sum}$ (6.3.10).

$$\delta_{\mathscr{P}}\left(\sum\right) = \cup\{|\delta_{\mathscr{P}^v}|\;|v \in \sum\}$$
$$\mathscr{P}_{\sum} = \{\sigma \in \mathscr{P} \text{ all the vertices of } \sigma\sigma \text{ are in } \mathscr{P}\}$$

$\mathscr{P}_{\sum}$ is full in $\mathscr{P}$ and $\delta_{\mathscr{P}}(\sum) = |N_{\mathscr{P}}(\mathscr{P}_{\sum})|$ is a regular neighbourhood of $|\mathscr{P}_{\sum}|$ in P.

Let $C(P) = P * v$ be a cone on P and $\varphi : C(P) \to [0,1]$ be the join of $v \to 0$ and $P \to 1$. If L is a subpolyhedron of P; $0 < \mathcal{L} < 1$, $C_{\mathcal{L}}(L)$ will **152**
mean $(L * v) \cap \varphi^{-1}([0,\mathcal{L}])$ as before. By $L \times [\alpha,\beta]$, $0 < \alpha < \beta < 1$, we shall mean $(L * v) \cap \varphi^{-1}([\alpha,\beta])$. In particular $C_{\mathcal{L}}(P) = \varphi^{-1}([0,\mathcal{L}])$ and $P \times [\alpha,\beta] = \varphi^{-1}([\alpha,\beta])$. The simplicial presentation $\mathscr{P} * \{\{v\}\}$ of $C(P)$ will be denoted by $C(\mathscr{P})$.

Proposition 6.8.3. *There is a centering of $C(\mathscr{P})$ with respect to which*

(1) $|\delta_{C(\varphi)}v| = C_{\frac{1}{2}}(P)$

(2) $|\delta_{C(\mathscr{P})}(a)| = |\delta_{\mathscr{P}}(a)| \times [\frac{1}{2},1]$ *for any vertex* $\mathfrak{a}$ *of* $\mathscr{P}$.

Proof. We take any centering η of $\mathscr{P}$, and extend it to $C(\ )$ by defining

$$\eta(\sigma\{v\}) = \frac{1}{2}\eta(\sigma) + \frac{1}{2}v, \quad \text{for} \quad \sigma \in \mathscr{P}.$$

Then it is obvious that φ is simplicial relative to $d(C(\mathscr{P}),\eta)$ and the triangulation of $[0,1]$ with vertices $\{0,\frac{1}{2},1\}$. From this it easily follows that $|\delta_{\mathscr{C}(\mathscr{P})^v}| = C_{\frac{1}{2}}(P)$.

The second assertion can be proved by a straight forward messy computation as follows:

A typical simplex of $\delta_{\mathscr{P}}(a)$ is a face of simplex of $d(\mathscr{P}, \eta)$ of the form $0(\eta_0, \eta_1, \ldots, \eta_k)$, with $a = \eta_0$, $\eta - i = \eta(\sigma_i)$, $\{a\} < \sigma_1 < \sigma_2 \cdots < \sigma_k, \sigma_i \in \mathscr{P}$. A point in $[\eta_0, \ldots, \eta_k] \times [\frac{1}{2}, 1]$ is uniquely determined by $t_0, \ldots, t_k$, $\mathcal{L}$, such that $t_i \geq 0$, $\sum_0^k t_i = 1$, $\frac{1}{2} \leq \mathcal{L} \leq 1$, and the point is:

$$(*) \qquad \alpha(\sum_0^k t_i \eta_i) + (1 - \alpha)v.$$

153 On the otherhand, a simplex of $\delta \subset (\mathscr{P})^{(a)}$ is a face of simplex determined by some ℓ between 0 and k, and vertices

$$\eta_0, \ldots, \eta_\ell, \quad \frac{1}{2}\eta_\ell + \frac{1}{2}v, \ldots, \frac{1}{2}\eta_k + \frac{1}{2}v,$$

with $a = \eta_0$, $\eta_i = \eta(\sigma_i)$, $\{a\} < \sigma_1 < \sigma_2 \ldots < \sigma_k$, $\sigma_i \in \mathscr{P}$. A typical point in the closure of such a simplex is uniquely determined by $r_0, \ldots, r_\ell$, $s_\ell, \ldots, s_k$, where r_i, $s_j \geq 0$ and $\sum\limits_{\circ}^{\ell} r_i + \sum\limits_{\ell}^{k} s_j = 1$. The point is

$$(**) \qquad \sum_0^\ell r_i \eta_i + \sum_\ell^k s_j (\frac{1}{2}\eta_j + \frac{1}{2}v).$$

Comparing coefficients in (*) and (**), we find that these points coincide if:

(A) $\alpha = 1 - \dfrac{1}{2} \sum_\ell^k s_j$

$t_i = \dfrac{r_i}{\alpha}, \ i < \ell$

$t = \dfrac{r_\ell + \frac{1}{2} s_\ell}{\alpha}$

$t_j = \dfrac{1}{2} \dfrac{s_j}{\alpha}, \ j > \ell$

(B) $r_i = \alpha t_i, \ i < \ell$

$r_\ell = \alpha(1 + \sum_\ell^k t_j) - 1$

$s_\ell = 2(1 - \alpha(1 + \sum_{\ell+1}^k t_j))$

$s_j = 2\alpha t_j, \ j > \ell.$

[To be sure, we should have started in (**) with an index different from k. But it can be easily seen that, when determing whether the points coincide, it is enough to consider (*) and (**)].

To show that $|\delta_{\mathscr{C}(\mathscr{P})}(a)| \subset |\delta_{\mathscr{P}}(a)| \times [\frac{1}{2}, 1]$, we need to check that if r's and s's satisfy their conditions (being ≥ 0, and of sum 1), then the solutions in (A) for α and the t's satisfy theirs ($\frac{1}{2} \leq \alpha \leq 1$ and the t's are ≥ 0 with sum 1). This is easy. **154**

To show that $|\delta_{\mathscr{P}}(a)| \times [\frac{1}{2}, 1] \subset |\delta_{\mathscr{C}(\mathscr{P})}(a)|$, we need to check if $\frac{1}{2} \leq \alpha \leq 1$, and the t's are ≥ 0 with sum 1, then there is some ℓ for which the solutions found in (B) satisfy the appropriate conditions. That the sum of r's and s's is one is clear; to make all ≥ 0, we take ℓ to be the maximum of those integers (m) for which

$$1 + \sum_{m}^{k} t_j \geq 1/\alpha$$

Since $1/\alpha \leq 2$, and $\sum_0^k t_j = 1$, there is such an ℓ; this choice of ℓ makes both r's and s's ≥ 0. □

Remark. If $\sum$ is a set of verticer of $\mathscr{P}$, we have as above,

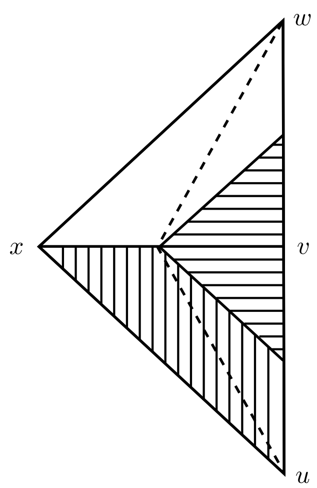
$$\delta_{\mathscr{C}(\mathscr{P})}(\sum) = \delta_{\mathscr{P}}(\sum) \times [\frac{1}{2}, 1].$$

Now let $P = X * \{u, w\}$ be a suspension. We define the *lower hemisphere* L of P to be $X * u$; it should be remarked that L is a regular neighbourhood of u in P.

Proposition 6.8.4. *With P, X, L as above there is a polyhedral equivalence $h : C(P) \to C(P)$ with $h|P = \mathrm{id}_P$, such that*

$$h\left(\left\{L \times \left[\frac{1}{2}, 1\right]\right\} \cup C_{\frac{1}{2}}(P)\right) = L \times \left[\frac{1}{2}, 1\right].$$

Proof. We can draw a picture which is a "cross section" through any particular point x in X:



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[The picture is actually the union of the two triangles $[x, v, w]$ and $[x, v, u]$ in $C(P)$, which we have flattened out to put in a planar picture. The vertically shaded part is the porition of $L \times [\frac{1}{2}, 1]$ in the cross section and the horizontally shaded part is the part of $C_{\frac{1}{2}}(P)$ in the cross section. We have to push the union of these two into the vertically shaded portion, and this uniformly over all cross sections].

From this picture we may see the following: $C(P)$ is the union of

$$A = \left\{X \times \left[\frac{1}{2}, 1\right]\right\} * \{u, w\}, \quad \text{and}$$

$$B = \left\{X \times \frac{1}{2}\right\} * J \quad \text{where} \quad J = v * \{u, w\}.$$

$$\text{And} \quad A \cap B = \left\{X \times \frac{1}{2}\right\} * \{u, w\}.$$

Now J is just, polyhedrally, an interval, and so there is obviously a polyhedral equivalence $f : J \to J$ such that

$$f(u) = u,\ f(w) = w$$

$$f(\frac{1}{2}v + \frac{1}{2}w) = \frac{1}{2}v + \frac{1}{2}u.$$

Such an f will take the part $[u, v] \cup [v, \frac{1}{2}v + \frac{1}{2}w]$ onto $[u, \frac{1}{2}u + \frac{1}{2}v]$.

Let $g : B \to B$ be the join of f on J and identity on $X \times \frac{1}{2}$. It is clear that $g|A \cap B$ is the identity map, and so by extending by identity on A, we get a polyhedral equivalence say $h : C(P) \to C(P)$. **156**

It should be pictorially evident that h has the desired properties. □

Putting all these together we get the proposition which we need:

Proposition 6.8.5. Hypotheses:

(1) *$\mathscr{P}$ is a simplicial presentation of P, $C(P)$ the cone over P with vertex v, $C(\mathscr{P}) = \mathscr{P} * \{\{v\}\}$, and $\sum$ a set of vertices of $\mathscr{P}$.*

(2) *There is a polyhedron X and a polyhedral equivalence $h : P \to X * \{u, w\}$ such that $h(|\mathscr{P}_{\sum}|) = Y * u$, for some $Y \subset X$.*

(3) *$|\delta_{\mathscr{C}(\mathscr{P})} v|$ and $\delta_{C(\mathscr{P})}(\sum)$ are constructed with reference to some centering $\mathscr{V}$ of $C(\mathscr{P})$.*

Conclusion: There is a polyhedral equivalence $\alpha : C(P) \to C(P)$ such that $\alpha|P = \mathrm{id}_P$ and α maps $(\delta_{C(\mathscr{P})}(\sum)) \cup (\delta_{\mathscr{C}(\mathscr{P})} v|$ onto $(\delta_{C(\mathscr{P})}(\sum))$.

Proof. Let η be the centering of $C(\mathscr{P})$ described in proposition 6.8.3. Let $f = f_{\eta, \mathscr{V}}$ be the simplicial isomorphism of $d(C(\mathscr{P}), \mathscr{V})$ onto $d(C(\mathscr{P}), \eta)$.

Let $h_1 : C(P) \to C(X * \{u, w\})$ be the join of $h : P \to X * \{u, w\}$ and the map vertex to vertex.

Now $\delta_{\mathscr{P}}(\sum)$ is a regular neighbourhood of $|\mathscr{P}_{\sum}|$ in P, and therefore $hf(\delta_{\mathscr{P}}(\sum))$ is a regular neighbourhood $hf(|\mathscr{P}_{\sum}|)$ in $X * \{u, w\}$. But $f(|\mathscr{P}_{\sum}|) = |\mathscr{P}_{\sum}|$ - infact f maps every $\mathscr{P}$-simplex onto itself - and $h(|\mathscr{P}_{\sum}|) = \gamma * u$. Thus $hf(\delta_{\mathscr{P}}(\sum))$ is a regular neighbourhood of $Y * u$ in **157**
$X * \{u, w\}$. Therefore by 6.8.2, $hf(\delta_{\mathscr{P}}(\sum))$ is a regular neighbourhood of u in $X * \{u, w\}$. But so is $X * u$. Hence there is a polyhedral equivalence $\beta : X * \{u, w\} \to X * \{u, w\}$ such that

$$\beta(hf(\delta_{\mathscr{P}}(\sum))) = X * u.$$

Let $\beta_1 : C(X * \{u, w\}) \to C(X * \{u, w\})$ be the join of and identity map of the vertex of the cone.

Now f is such that $f(\delta_{C(\mathscr{P})}(\sum)) = f(\delta_{\mathscr{P}}(\sum)) \times [\frac{1}{2}, 1]$ and $f(|\delta_{C(\mathscr{P})} v|) = C_{\frac{1}{2}}(P)$.

Since β_1 and h_1 are radial extensions the same thing holds, i.e.

$$\begin{aligned}\beta_1 h_1 f(\delta_{(C|\mathscr{P})}(\sum)) &= \beta_1 h_1 (f(\delta_{\mathscr{P}}(\sum)) \times [\frac{1}{2}, 1] \\ &= \beta h f(\delta_{\mathscr{P}}(\sum)) \times [\frac{1}{2}, 1]\end{aligned}$$

which is $\{X * u\} \times [\frac{1}{2}, 1]$, and

$$\begin{aligned}\beta_1 h_1 f(|\delta_{C(\mathscr{P})} v|) &= \beta_1 h_1 (C_{\frac{1}{2}}(P)) \\ &= C_{\frac{1}{2}}(X * \{u, w\}).\end{aligned}$$

Applying 6.8.4, we get a polyhedral equivalence

$$\gamma : C(X * \{u, w\}) \to C(X * \{u, w\}) \quad \text{with} \quad \gamma | X * \{u, w\} =$$

identity and

$$\begin{aligned}\gamma\left((X * u) \times [\frac{1}{2}, 1] \cup C_{\frac{1}{2}}(X * \{u, w\})\right) \\ = (X * u) \times [\frac{1}{2}, 1].\end{aligned}$$

The desired map α is now,

$$\alpha = f^{-1} \circ h_1^{-1} \circ \beta_1^{-1} \circ \gamma \circ \beta_1 \circ h_1 \circ f.$$

$\square$

158 We will now write down two specific corollaries of proposition 6.8.5, which will immediately give the regular neighbourhood theorem. First we recall the notation at the end of section 3 (6.3.10).

If $\mathscr{P}$ is regular presentation, given a centering η of $\mathscr{P}$ and a centering of $d(\mathscr{P}, \eta)$, we defined

$$C^* = |\delta_{d\mathscr{P}}(\eta C)|, \quad \text{for any} \quad C$$

and $\mathscr{N}^* = \cup\{C^* | C \in \mathscr{N}\}$, for any subset $\mathscr{N}$ of $\mathscr{P}$. If $\mathscr{N}$ is a subpresentation of $\mathscr{P}$, then $d(\mathscr{N},\eta)$ (where $\eta|\mathscr{N}$ is again denoted by η) is full in $d(\mathscr{P},\eta)$. Writing $\mathscr{N}' = d(\mathscr{N},\eta)\mathscr{P}' = d(\mathscr{P},\eta)$, and $\sum$ as the set of vertices of $\mathscr{P}'$ of the form ηC, for $C \in \mathscr{N}$, we see that $\mathscr{P}'_{\sum} = \mathscr{N}'$ and $\delta_{\mathscr{P}'}(\sum) = \mathscr{N}^*$, which is a regular neighbourhood of $|\mathscr{N}|$ in $|\mathscr{P}|$.

Corollary 6.8.6. *Let $\mathscr{P}$ be a regular presentation with a subpresentation $\mathscr{N}$, E a free edge of $\mathscr{N}$ with attaching membrane A such that (E, A) is homogeneous in $\mathscr{P}$. Then there is a polyhedral equivalence $h = |\mathscr{P}| \to |\mathscr{P}|$ which is identity outside of $\overline{E} * |\lambda_{\mathscr{P}} E|$ and which takes $\mathscr{N}^*$ onto $(\mathscr{N} - \{E\})^*$.*

[Note: It is understood that there is a centering η of $\mathscr{P}$, and a centering of $d(\mathscr{P},\eta)$].

Proof. Look at $St(\eta E, d\mathscr{P})$; this is a presentation say $\mathscr{P}'$ of $\overline{E} * |\lambda_{\mathscr{P}} E|$. Let $\sum$ denote the set of vertices of $d\mathscr{P}$ of the form ηF for $F < E$ and ηA. Then $|\mathscr{P}'_{\sum}|$ is the join of ∂E to ηA. Since $|\lambda_{\mathscr{P}} E|$ is equivalent to a
suspension (homogenity of (E, A)) with ηA going to a pole, we see that **159**
$\partial E * |\lambda_{\mathscr{P}} E|$ is equivalent to a suspension with $|\mathscr{P}'_{\sum}|$ going to a subcone. And $\overline{E} * |\lambda_{\mathscr{P}} E|$ is a cone over $\partial E * |\lambda_{\mathscr{P}} E|$. And consider the centering of $\mathscr{P}'$ coming from that of $d(\mathscr{P},\eta)$.

Thus we have the situation of 6.8.5, and making the necessary substitutions in 6.8.5, we get a polyhedral equivalence α of $|\mathscr{P}'| = \overline{E} * |\lambda_{\mathscr{P}} E|$ taking $|\delta'_{\mathscr{P}}(\eta E)| \cup \delta_{\mathscr{P}'}(\sum)$ onto $\delta_{\mathscr{P}'}(\sum)$. $|\delta_{\mathscr{P}'}(\eta E)|$ is just E^*. Now observe that the set of centres of elements of $\mathscr{N}$ in $\mathscr{P}'$ is $\sum \cup \{\eta E\}$. (This is where we use the fact that E is a free edge). Therefore $\mathscr{N}^* \cap |\mathscr{P}'| = E^* \cup \delta_{\mathscr{P}'}(\sum)$ and $(\mathscr{N} - \{E\})^* \cap |\mathscr{P}'| = \delta_{\mathscr{P}'}(\sum)$. So α takes the part of $(\mathscr{N}^*)$ in $|\mathscr{P}'|$ onto the part of $(\mathscr{N} - \{E\})^*$ in $|\mathscr{P}'|$. α is identity on the base of the cone, and $E^* \subset |\mathscr{P}'|$. Therefore extending α to an equivalence h of $|\mathscr{P}|$ by patching up with identity outside $|\mathscr{P}'|$, we see that h takes $\mathscr{N}^*$ onto $(\mathscr{N} - \{E\})^*$ and is identity outside $|\mathscr{P}'| = \overline{E} * |\lambda_{\mathscr{P}} E|$. □

Corollary 6.8.7. *In the same situation, there is a polyhedral equivalence $h' : |\mathscr{P}| \to |\mathscr{P}|$ which is identity outside $\overline{A} * |\lambda_{\mathscr{P}} A|$, which takes $(\mathscr{N} - \{E\})^*$ onto $(\mathscr{N} - \{E, A\})^*$.*

[for this corollary we need only that E is a free edge of A, and A is the attaching membrane. Homogenity of (E, A) is not necessary].

Proof. This time we call $\mathscr{P}' = St(\eta A, d\mathscr{P})$, and $\sum$ the set of vertices ηF, $F < A$ and $F \neq E$. Then $|\mathscr{P}'_{\sum}| = \partial A - E$. ∂A is equivalent to a
160 suspension with $\partial A - E$ as the lower hemisphere. Hence $\partial A * |\lambda_{\mathscr{P}} A|$ is equivalent to a suspension with $|\mathscr{P}'_{\sum}|$ mapping onto a subcone. Applying 6.8.5, we get a polyhedral equivalence α' of $\overline{A} * |\lambda_{\mathscr{P}} A|$ on itself, which is identity on $\partial A * |\lambda_{\mathscr{P}} A|$ and takes $\delta_{\mathscr{P}'}(\sum) \cup A^*$ onto $\delta_{\mathscr{P}'}(\sum)$. Since E is a free edge and A is principal in $\mathscr{N}$, $\delta_{\mathscr{P}'}(\sum)$ is just the part of $(\mathscr{N} - \{E, A\})^*$ in $\overline{A} * |\lambda_{\mathscr{P}} A|$, and $\delta_{\mathscr{P}'}(\sum) \cup A^*$ is the part of $(\mathscr{N} - \{E\})^*$ in $\overline{A} * |\lambda_{\mathscr{P}} A|$. Extending α' to an equivalence h' of $|\mathscr{P}|$ by patching up with identity outside $\overline{A} * |\lambda_{\mathscr{P}} A|$, since A^* is contained in $\overline{A} * |\lambda_{\mathscr{P}} A|$ we see that h' takes $(\mathscr{N} - \{E\})^*$ onto $(\mathscr{N} - \{E, A\})^*$ and is identity outside $\overline{A} * |\lambda_{\mathscr{P}} A|$. $\square$

Thus in the situation of 6.8.6, if we take the composition $h'h$ of the equivalences given by 6.8.6 and 6.8.7, $h' \circ h$ takes $\mathscr{N}^*$ onto $(\mathscr{N} - \{E, A\})^*$. Support of $h' \subset \overline{A} * |\lambda_{\mathscr{P}} A|$, support of $h \subset \overline{E} * |\lambda_{\mathscr{P}} E|$, hence $h' \circ h$ fixes, the polyhedron $|(\mathscr{N} - \{E, A\})|$. This at once gives,

Proposition 6.8.8. *If $\mathscr{N} \searrow \mathscr{X}$ homogeneously in $\mathscr{P}$, then there is a polyhedral equivalence of $|\mathscr{P}|$, which is identity on $|\mathscr{X}|$ and takes $\mathscr{N}^*$ onto $\mathscr{X}^*$.*

Corollary 6.8.9. *If $N \searrow X$ homogeneously in P, then any regular neighbourhood of N in P is a regular neighbourhood of X in P.*

Proof of the regular neighbourhood theorem 6.8.1.

By 6.8.9 any regular neighbourhood say N' of N is a regular neighbourhood of X. Since N is bicollared in P, there is a polyhedral equivalence h of P taking N onto N'. Since $X \subset \mathrm{Int}_P N$, h can be chosen to be fixed on X (see 6.4.8). Therefore N is a regular neighbourhood of X.

6.9 Some applications and remarks

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In this section we make a few observations about the previous concepts in the context of PL-manifolds

6.9.1 Let M be a PL-manifold, ∂M its boundary, $\mathscr{P}$ a regular presentation of M. Let $E, A \in \mathscr{P}$, $E < A$ and $\dim A = \dim E + 1$. (E, A) is homogeneous in $\mathscr{P}$ if and only if either both E and A are in ∂M or both E and A are in $M - \partial M$.

Proof. Let η be a centering of $\mathscr{P}$. Let $E' \subset E$ a simplex of $d(\mathscr{P}, \eta)$ of dimension $= \dim E$, and $A' = \{\eta A\}E'$. Now the problem is equivalent to: When is $|LK(E', d\mathscr{P})|$ equivalent to a suspension with ηA going to a vertex? If E and A are in $M - \partial M$, so are E' and A' and $|LK(E', d\mathscr{P})|$ is a sphere, hence it is possible. If E and A are both in ∂M, so are E' and A' and $|Lk(E', d\mathscr{P})|$ is a cell, with ηA contained in the boundary. So again it is possible. If E is in ∂M and A is in $M - \partial M$ so are E' and A' and $|Lk(E', d\mathscr{P})|$ is a cell with ηA in the interior. Hence in this case it is impossible. □

Suppose now that $\mathscr{N}$ and $\mathscr{X}$ are subpresentation of $\mathscr{P}$ and $\mathscr{N} \searrow \mathscr{X}$ homogeneously in $\mathscr{P}$. In the sequence of (elementary) homogeneous of collapses from $\mathscr{N}$ to $\mathscr{X}$, if a collapse C_1 in the boundary comes before a collapse C_2 in the interior we can interchange them i.e. if $\mathscr{N}_{i-1} \searrow^{C_1} \mathscr{N}_i \searrow^{C_2} \mathscr{N}_{i+1}$, then we can find $\mathscr{N}_i'$ such that $\mathscr{N}_{i-1} \searrow^{C_2'} \mathscr{N}_i' \searrow^{C_1'} \mathscr{N}_{i+1}$ and the free edge and attaching membrane of C_i and C_i', $j = 1, 2$ are the same. Doing this a finite number of times we have

6.9.2 If $N \searrow X$ homogeneously in M, then $N \searrow X \cup (N \cap \partial M) \searrow X$. In 162
particular, this is true for regular neighbourhoods. Some rearrangement is possible for the usual elementary collapses also:

Ex. 6.9.3. Suppose $\mathscr{P} \searrow Q$, combinatorially, and $\mathscr{P}_1, \ldots, \mathscr{P}_k$, $1 \le i \le k$ are subpresentations such that $\mathscr{P}_i$ is obtained from $\mathscr{P}_{i-1}$ by an elementary collapse at the free edge E_{i-1} with attaching membrane A_{i-1} and $\mathscr{P} = \mathscr{P}_1$, $Q = \mathscr{P}_k$. Then we can find subpresentations $\mathscr{P}_1'$, $\mathscr{P}_2', \ldots, \mathscr{P}_k'$, $\mathscr{P}_1' = \mathscr{P}$, $\mathscr{P}_k' = Q$, such that $\mathscr{P}_i'$ is obtained from $\mathscr{P}_{i-1}'$ by an elementary collapse at the free edge E_{i-1}' with attaching membrane A_{i-1}' and $\dim A_i' \ge \dim A_{i-1}'$. Moreover, except for order, the pairs (E_i', A_i') are the same as the pairs (E_j, A_j).

More briefly, we can rearrange the collapses in the order of non-increasing dimension.

Ex. 6.9.4. An n-cell collapses to any $(n-1)$-cell in its boundary. This follows from 6.5.10.

Ex. 6.9.5. An n-cell is collapsible to any point in it.

We call polyhedron *collapsible* if it collapses to a point.

Ex. 6.9.5′. A collapsible polyhedron collapses to any point in it.

[Hint: By virtue of 6.9.3, it is enough to consider one dimensional collapsible presentations with the given point as a vertex].

6.9.6 If M is a collapsible PL n-manifold, then M is a n-cell.

Sketch of the proof: $\partial M \neq \emptyset$, for if $\partial M = \emptyset$, there is no free edge to start the collapsing. Next we can assume that M collapses to a point in
163 $M - \partial M$, either by 6.9.5′ or by 6.9.4 and 6.5.11. Now attach a collar of ∂M to M (to get PL-manifold M') so that all the collapsing is in the interior of M', hence homogeneous. Now all the conditions of the regular neighbourhood theorem are satisfied. Hence M is the regular neighbourhood of a print in M', hence an n-cell.

The following two remarks will be useful in the next chapter.

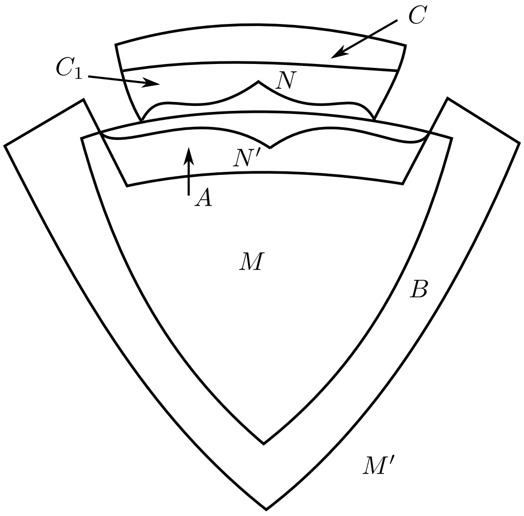
6.9.7 Let $f : K \times D^{n-k} \to M^n$ be an imbedding into int M, where K is a K-manifold and D^{n-k} an $(n-k)$-cell. Then $f(K \times D^{n-k})$ can be shrunk into any given neighbourhood of $f(K \times e)$ in M, for a fixed $e \in \operatorname{int} D^{n-k}$ by an isotopy which can be assumed to be fixed on $f(K \times e)$.

$K \times D^{n-k} \searrow K \times e$ (this follows, for example, from 6.5.14 by induction). It is easily seen that $f(K \times D^{n-k})$ is a neighbourhood of $f(k \times e)$ in M and is bicollared.

Proposition 6.9.8. *Let M be a PL n-manifold, and N a PL $(n-1)$-manifold in ∂M, and $M \searrow N$. Then M is polyhedrally equivalent to $N \times I$. Moreover the polyhedral equivalence $h : M \approx N \times I$, can be so chosen that $h(n) = (n = 0)$ for $n \in N$.*

Proof. Such an N cannot be the whole of ∂M. Either $\partial N \neq \emptyset$, or N is a finite union of components of ∂M (see 4.4.16). In any case N is bicollared in ∂M. If N' is regular neighbourhood of N in ∂M, since N is bicollared in ∂M, N' is polyhedrally equivalent to N (6.4.8).

Since $M \searrow N$, there is a regular neighbourhood say A of N in M such that $M \searrow A$ (see 6.6.9). Let $A \cap \partial M = N'$. Then N' is a regular neighbourhood of N in ∂M. It is clear that A is polyhedrally equivalent **164**
to $N \times I$. Now attach B and C, B a collar over $\overline{\partial M - N'}$ and C a collar over N to M such that $B \cap C = \emptyset$. Let the resulting manifold be M'.



Consider another collar $C_1 \subset C$, and the manifolds $A \cup C_1$ and $M \cup C_1$. In M', all the collapses from M to A are in the interior and hence $M \searrow A$ homogeneously in M', and the collapsing from $A \searrow N$ continues to be homogeneous in M. Clearly $C_1 \searrow N$ homogeneously in M'. Thus both $A \cup C_1$ and $M \cup C_1$ collapse homogeneously in M' to N, both are neighbourhoods of N in M' and both are bicollared. Hence there is an equivalence $A \cup C_1 \approx M \cup C_1$. Clearly $A \cup C_1 \approx N \times I$. Hence $M \cup C_1 \approx N \times I$, hence $M \approx N \times I$.

To prove the last remark observe that if $\mathscr{L} : N \times I \approx C_1$ is an equivalence such that $\mathscr{L}(n, 1) = n$, for $n \in N$, the equivalence $M \approx$

$M \cup C_1$ can be chosen such that it carries $n \in N$ to $\mathscr{L}(n, 0) \in C_1$. Finally the equivalence $A \cup C_1 \approx M \cup C_1$ can be assumed to be identity on C_1. □

6.10 Conclusion

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Now let us, recaptitulate briefly the programme for proving the regular neighbourhood theorem:

(A) We have a notion of equivalence of pairs

$$(P, X) \approx (P', X')$$

(B) We define a regular neighbourhood of X in P to be any thing equivalent by an auto-equivalence of (P, X) to $|N_{\mathscr{P}}(\mathscr{X})|$, where $\mathscr{P}$ is a simplicial presentation of P with a full subpresentation $\mathscr{X}$ covering X.

(C) We have the notions of the cone on P, suspension on P, and $P \times I$; and hence the idea of local collaring, collaring and bicollaring.

(D) We can prove: We can prove: $P \times [0, \frac{1}{2}]$ is a regular neighbourhood of $P \times 0$ in $P \times [0, 1]$. The lower half of the suspension of P is a regular neighbourhood of a pole. A locally collared subpolyhedron is collared. Regular neighbourhood of a pole. A locally collared subpolyhedron is collared. Regular neighbourhoods are bicollared.

(E) We have for regular presentations, the notion of collapsing, and of homogeneous; and we prove that $N \searrow X$ homogeneously in P if N is a regular neighbourhood of X in P.

(F) Finally, we prove the converse, that if $N \searrow X$ homogeneously in P, then a regular neighbourhood of N is a regular neighbourhood of X. We pick up a particular regular neighbourhood of N and strink it down a bit at a time to a particular regular neighbourhood of X. In doing this, we need to have proved the theorem for a

166 particular case: X' is a pole of a suspension P' and N' is a subcone of P'. An analysis of the proof shows that we need the result for various P' of dimension less than that of P. Hence we could have proved this by induction on dimension, although it is simple enough to prove in the special case by construction.

Now it should be remarked that precisely the same programme can be carried out in other contexts. In particular for pairs:

A pair (P, Q) is a polyhedron P with a subpolyhedron Q; we say $(P_1, Q_1) \subset (P_2, Q_2)$ if $P_1 \subset P_2$, and $Q_1 = Q_2 \cap P_1$. If $(P_1, Q_1) \subset (P_2, Q_2)$ we define the boundary of the former in the latter to be $(bd_{P_2}P_1, Q_1 \cap bd_{P_2}P_1)$.

Define an equivalence $h : (P_1, Q_1) \to (P_2, Q_2)$ to be a polyhedral equivalence $\alpha : P_1 \approx P_2$ mapping Q_1 onto Q_2.

An admissible presentation of (P, Q) is a pair of regular presentations $\mathcal{Q} \subset \mathscr{P}$ with $|\mathscr{P}| = P$, $|\mathcal{Q}| = Q$. A free edge of an admissible presentation $(\mathscr{P}, \mathcal{Q})$ is an $E \in \mathscr{P}$, which is a free edge of $\mathscr{P}$ with attaching membrane A, such that if $E \in \mathcal{Q}$, then $A \in \mathcal{Q}$.

The programme can be carried out mechanically with the obvious definition of homogeneous collapsing.

Finally, we draw some consequences, by applying to PL-manifolds.

Let $A \subset B$, where A is a PL a-manifold and B is a PL b-manifold. We say (B, A) is locally *un-knotted* if, for every $x \in A$, if (L_B, L_A) is polyhedrally equivalent to $(L_A * X, L_A)$ for some X. It is possible to 167
show that X must be either a cell or a sphere of dimension $b - a - 1$; and that if A is connected, then either all the X's are cells, in which case A is locally un-knotted in ∂B or all X's are spheres, in which case $\partial A = A \cap \partial B$.

It then occurs as in the case of a single manifold, that all the collapsing (in the pair sense) which is in the interior of (B, A) is homogeneous, and hence we can prove the following result:

Let $D^a \subset \Delta^b$, with (Δ, D) a locally un-knotted pair of the sort where $\partial D = D \cap \partial\Delta$. Then if $\Delta \searrow D \searrow$ point, the pair (Δ, D) is an absolute regular neighbourhood of a point (relative to $(\partial\Delta, \partial D)$ and so (Δ, D) is polyhedrally equivalent to $(S * D, D)$ where S is a $(b - a - 1)$-sphere, i.e. (Δ, D) is un-knotted.

[This is a key lemma for Zeeman's theorem, that $(b - a) \geq 3 \Rightarrow (\Delta, D)$ is un-knotted. See Zemman "Seminar on combinatorial Topology", Chapter IV, pp. 4-5].

Chapter 7

Regular collapsing and applications

7.1 Regular collapsing

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Let $\mathscr{S}$ be a simplicial presentation. We say that $\sigma \in \mathscr{S}$ is an *outer edge* of $\mathscr{S}$, if there is a $\Delta \in \mathscr{S}$, such that if $\sigma \leq \rho$, $\rho \in \mathscr{S}$, then $\rho \leq \Delta$, and $\dim \Delta > \dim \sigma$. In this case Δ is uniquely fixed by σ, and is of the form $\Delta = \sigma\tau$, $\tau \neq \emptyset$. The elements of $\mathscr{S}$ having σ as a face are exactly of the form $\sigma\tau'$, $\tau' \leq \tau$. The remaining faces of Δ are of the form $\sigma'\tau'$, $\sigma' < \sigma$, $\tau' \leq \tau$; in otherwords they consist of $\{\partial\sigma\} * \{\overline{\tau}\}$. Thus

$$\mathscr{S}' = \mathscr{S} - \{\overline{\Delta}\} \cup [\{\partial\sigma\} * \{\overline{\tau}\}]$$

is a subpresentation of $\mathscr{S}$, and

$$|\mathscr{S}| = |\mathscr{S}'| \cup \overline{\Delta}$$
$$|\mathscr{S}'| \cap \overline{\Delta} = \partial\sigma * \overline{\tau}$$

Let dim $= \Delta = n$. Then, we say that $\mathscr{S}'$ is obtained from $\mathscr{S}$ by an *elementary regular collapse* (n) with outer edge σ and *major simplex* Δ.

If $\mathscr{S} = \mathscr{S}_1, \ldots, \mathscr{S}_k = \mathscr{Z}$, and $\mathscr{S}_{i+1}$ is obtained from $\mathscr{S}_i$ by an elementary regular collapse (n), we say that $\mathscr{S}$ *regularly collapses* (n) to $\mathscr{Z}$.

The elements of the theory of regular collapsing can be approached from the point of view of "steller subdivisions" (cf. Section 13 of "simplicial spaces, nuclei and *m*-groups" or the first few pages of Zeeman's "unknotting spheres", Annals of Mathematics, 72, (1960) 350-361), but for the sake of novelty we shall to something else.

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7.1.1 Recalling Notations. $\sigma, \tau, \ldots$ usually denote open simplexes. $\overline{\sigma}$, $\overline{\tau}, \ldots$ denote their closures (closed simplexes), and $\partial\sigma, \partial\tau, \ldots$ their boundaries. The simplicial presentation of $\overline{\sigma}$ consisting of σ and its faces in denoted by $\{\overline{\sigma}\}$, and that of $\partial\sigma$ consisting of faces of σ by $\{\partial\sigma\}$ (see 54). $\sigma\tau$ stands for the join of the two open simplexes σ and τ, when the join is defined. If σ is a 0-simplex and x is the unique point of σ, we will write $\{x\}\tau$ for $\sigma\tau$. On the other hand the join of two polyhedra P and Q when it is defined is denoted by $P * Q$. Similarly the join of two simplicial presentations $\mathscr{P}$ and $\mathcal{Q}$ when it is defined is denoted by $\mathscr{P} * \mathcal{Q}$. For example if $\sigma\tau$ is defined, then $\{\partial\sigma\} * \{\overline{\tau}\}$ is the canonical simplicial presentation of the polyhedron $\partial\sigma * \overline{\tau}$. If P is a polyhedron consisting of a single point x, we will sometimes write $x * Q$ instead of $P * Q$. With this notation $\overline{\{x\}\sigma}$ and $x * \overline{\sigma}$ are the same.

Let Δ be an $(n-1)$-simplex, $I = [0, 1]$, and let $\mathscr{S}$ be a simplicial presentation of $\overline{\Delta} \times I$ such that the projection $p : \overline{\Delta} \times I \to \overline{\Delta}$ is simplicial with reference to $\mathscr{S}$ and $\{\overline{\Delta}\}$.

The n-simplexes of $\mathscr{S}$ can be ordered as follows: $\Gamma_1, \ldots, \Gamma_k$, so that if $x \in \Delta$, $x \times I$ intersects the Γ_i's in order. That is $\Delta \times 0$ is a face of Γ_1, Γ_1 has another face Δ_1 which maps onto Δ, Δ_1 is a face of $\Gamma_2, \ldots, \Delta_{i-1}$ is a face of Γ_i, but Γ_i has another face that maps onto Δ, call it Δ_i and so on. We start with $\Delta_0 = \Delta \times 0$ and end up with $\Delta_k = \Delta \times 1$.

170 Let us write $\Delta = \sigma\tau$ in some way. Let $T = \overline{\Delta} \times 0 \cup (\partial\sigma * \overline{\tau}) \times I$. (If $\sigma = \emptyset$, T should be taken to be just $\overline{\Delta} \times 0$). Then there is a subpresentation $\mathscr{L}$ of $\mathscr{S}$ which covers T.

Lemma 7.1.2. *(With the above hypotheses and notation) $\mathscr{S}$ regularly collapses (n) to $\mathscr{L}$.*

Proof. We in fact show that there is a sequence of regular collapses with major simplexes $\Gamma_k, \ldots, \Gamma_1$. We must then define

$$\mathscr{S}_i = \mathscr{Z} \cup \{\overline{\Gamma}_1\} \cup \ldots \cup \{\overline{\Gamma}_i\}$$

and find some outer edge lying on $\overline{\Gamma}_i$, so that the corresponding regular collapse results in $\mathscr{S}_{i-1}$.

Now Γ_i is an n-simplex and its projection Δ is an $(n-1)$-simplex, therefore there are two vertices v_1 and v_2 of Γ_i (choose v_1, v_2 so that the I-co-ordinate of v_1 is < the I-co-ordinate of v_2) which map into one vertex v of Δ. Now $\Delta = \sigma\tau$ and so v is a vertex of either σ or τ.

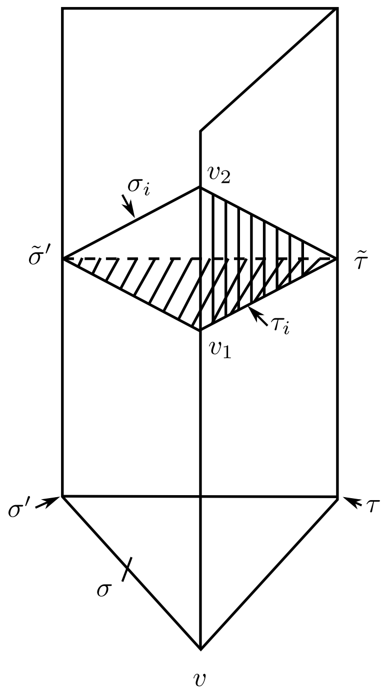
Case 1: v is a vertex of σ. Write $\sigma = \{v\}\sigma'$. Let $\overline{\sigma}'$ and $\overline{\tau}$ be the faces of Γ_i lying over σ' and τ. Then

$$\Gamma_i = \{v_1\}\{v_2\}\tilde{\sigma}'\tilde{\tau}$$

and the two faces of Γ_i which are mapped onto Δ are

$$\begin{aligned} &\{v_1\}\tilde{\sigma}'\tilde{\tau} = \Delta_{i-1} \\ \text{and} \qquad &\{v_2\}\tilde{\sigma}'\tilde{\tau} = \Delta_i. \end{aligned}$$

Define $\sigma_i = \{v_2\}\tilde{\sigma}'$, $\tau_i = \{v_1\}\tilde{\tau}$. It is claimed that if we take σ_i as an **171**
outer edge then the result of the elementary regular collapse with major simplex Γ_i is $\mathscr{S}_{i-1}$.



$\square$

σ_i cannot be in $\mathscr{L}$; because the only ($\dim\sigma$)-simplex in $\sigma\times I$ which is in $\mathscr{L}$ is σ, and $\sigma_i \neq \sigma$ since v_2 is a vertex of σ_i. Also Γ_i is the only simplex among $\Gamma_1,\ldots,\Gamma_i$ which contains v_2 as a vertex. Hence if $\sigma_i \leq \rho$, $\rho \in \mathscr{S}_i$, then $\rho \leq \Gamma_i$.

We then have to show that $\overline{\Gamma}_i \cap \mathscr{S}_{i-1} = \partial\sigma_i * \tau_i$

$$\begin{aligned}\partial\sigma_i * \overline{\tau}_i &= \partial(\{v_2\}\tilde{\sigma}') * (\overline{\{v_1\}\tilde{\tau}})\\ &= (\overline{\tilde{\sigma}'\{v_1\}\tilde{\tau}}) \cup (v_2 * \partial\tilde{\sigma}' * \overline{\{v_1\}\tilde{\tau}}).\end{aligned}$$

The first term here is $\overline{\Delta}_{i-1}$, which is where $\overline{\Gamma}_i$ intersects $\overline{\Gamma}_1\cup\ldots\cup\overline{\Gamma}_{i-1}$.
172 The second term written slightly differently is $[v_1v_2] * \partial\tilde{\sigma}' * \overline{\tilde{\tau}}$ to which we may add a part of the first term namely $(\overline{\tilde{\sigma}'\tilde{\tau}})$ to obtain all faces of Γ_i which map to

$$\partial\sigma * \overline{\tau} = \partial(\{v\}\sigma^1) * \overline{\tau}$$

$$= (\overline{\sigma}' * \overline{\tau}) \cup [\partial\sigma' * \overline{\tau} * v]$$

In other words, this is $\overline{\Gamma}_i \cap [(\partial\sigma * \overline{\tau}) \times I]$.

This shows that

$$\overline{\Gamma}_i \cap |\mathscr{S}_{i-1}| = \partial\sigma_i * \overline{\tau}_i;$$

and so $\mathscr{S}_i$ to $\mathscr{S}_{i-1}$ is an elementary regular collapse with outer edge σ_i and major simplex Γ_i.

Case 2: v is a vertex of τ. Wrtie $\tau = v\tau'$, define $\tilde{\sigma}$, $\tilde{\sigma}'$ to be faces of Γ_i lying over σ and τ'. In this case

$$\Gamma_i = \{v_1\}\{v_2\}\tilde{\sigma}\tilde{\tau}'$$

and the two faces of Γ_i which are mapped on Δ are

$$\begin{aligned} &\{v_1\}\tilde{\sigma}\tilde{\sigma}' = \Delta_{i-1} \\ \text{and} \qquad &\{v_2\}\tilde{\sigma}\tilde{\tau}' = \Delta_i. \end{aligned}$$

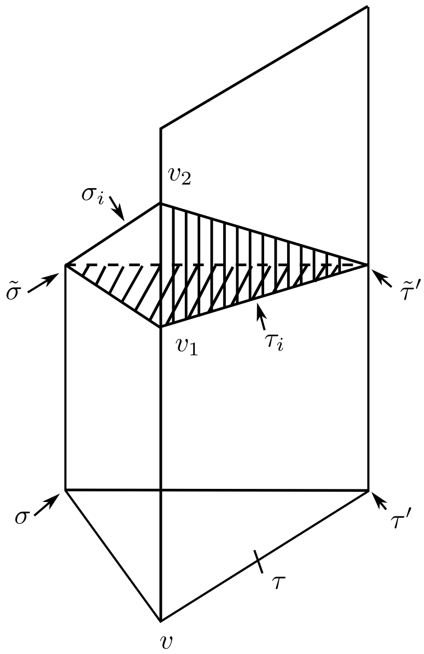
We now define

$$\begin{aligned} \sigma_i &= \{v_2\}\tilde{\sigma} \\ \tau_i &= \{v_1\}\tilde{\tau}' \end{aligned}$$

and make computations as before.

$$\begin{aligned} &\partial\sigma_i * \overline{\tau}_i \\ &= (\overline{\tilde{\sigma}} * \overline{\{v_1\}\tilde{\tau}'}) \cup (v_2 * \partial\tilde{\sigma} * \overline{\{v_1\}\tilde{\tau}'}) \\ &= \overline{\Delta}_{i-1} \cup \partial\tilde{\sigma} * [v_1 v_2] * \overline{\tilde{\tau}'} \end{aligned}$$

and $\partial\tilde{\sigma} * [v_1 v_2] * \overline{\tilde{\tau}'} = \overline{\Gamma}_i \cap [(\partial\sigma * \overline{\tau}) \times I]$



And this shows that if we perform an elementary regular collapse (n) on $\mathscr{S}_i$ with outer edge σ_i and major simplex Γ_i, we get $\mathscr{S}_{i-1}$.

Hence $\mathscr{S}$ regularly collapses (n) to $\mathscr{Z}$.

Define $I^1 = I$, $I^k = I^{k-1} \times I$, $T_1 = 0 \subset I'$, and $T_k = (I^{k-1} \times 0) \cup (T_{k-1} \times I) \subset I^k$.

It is easy to see that T_k is a $(k-1)$-cell in ∂I^k, and is the set of points of I^k at least one co-ordinate of which is zero.

174 Let $\alpha_k : I^k = I^{k-1} \times I \to I^{k-1}$ be the projection.

Lemma 7.1.3. *Let $\mathscr{S}_n, \mathscr{S}_{n-1}, \ldots, \mathscr{S}_1$ be simplicial presentations of I^n, $I^{n-1}, \ldots, I^1$ with respect to which all the maps $\alpha_n, \ldots, \alpha_2$, are simplicial. Then there exist subpresentations $\mathscr{Z}_n$, $\mathscr{Z}_{n-1}, \ldots, \mathscr{Z}_1$ covering T_n, $T_{n-1}, \ldots, T_1$ respectively, and such that $\mathscr{S}_i$ regularly collapses (i) to $\mathscr{Z}_i$ for all i.*

Proof. The proof is by induction. It is easily verified that $\mathscr{S}_1$ collapses (1) to $\mathscr{Z}_1$.

So, inductively, we know that $\mathscr{S}_i$ collapses (i) to $\mathscr{Z}_i$, for $i \leq n-1$. Now $\mathscr{Z}_n$ is just the subpresentation of $\mathscr{S}_n$ covering $I^{n-1} \times 0 \cup |\mathscr{Z}_{n-1}| \times I = T_n$.

Let the collapsing of $\mathscr{S}_{n-1}$ to $\mathscr{Z}_{n-1}$ occur along the major simplexes $\Delta_1, \ldots, \Delta_k$. Then we define

$$\mathfrak{a}_i = \mathscr{Z}_{n-1} \cup \{\overline{\Delta}_i\} \ldots \{\overline{\Delta}_k\}$$

and write $\Delta_i = \sigma_i \tau_i$, where σ_i is the outer edge of the regular collapse $(n-1)$ from $\mathfrak{a}_i$ to $\mathfrak{a}_{i+1}$. Then $\overline{\Delta}_i \cap |\mathfrak{a}_{i+1}| = \partial\sigma_i * \overline{\tau}_i$.

Define $\mathscr{B}_i =$ the subpresentation of $\mathscr{S}_n$ covering $I^{n-1} \times 0$ plus α_n^{-1} $(|\mathfrak{a}_i|)$. Thus $\mathscr{B}_1 = \mathscr{S}_n$ and $\mathscr{B}_{k+1} = \mathscr{Z}_n$.

We will show that $\mathscr{B}_i$ regularly collapses (n) to $\mathscr{B}_{i+1}$, stringing these together, then $\mathscr{S}_n$ regularly collapses (n) to $\mathscr{Z}_n$.

To show that $\mathscr{B}_i$ regularly collapses (n) to $\mathscr{B}_{i+1}$ it is enough to look 175
at the part of $\mathscr{B}_i$ covering $\alpha_n^{-1}(\overline{\Delta}_i)$ i.e. $\overline{\Delta}_i \times I$. $\overline{\Delta}_i \times I \cap |\mathscr{B}_{i+1}| = \overline{\Delta}_i \times 0 \cup [(\partial\sigma_i * \overline{\tau}_i) \times I]$ and $\alpha_n | \overline{\Delta}_i \times I$ is just the projection $\overline{\Delta}_i \times I \to \overline{\Delta}_i$ which is simplicial with reference to the subpresentation of $\mathscr{S}_n$ covering $\overline{\Delta}_i \times I$ and $\{\overline{\Delta}_i\}$. And our lemma 7.1.2 is especially tailored for this situation. □

Theorem 7.1.4. *Let A be a n-cell, B an n-cell in A, and $\mathscr{P}$ a regular presentation of A. Then there is a simplicial presentation $\mathscr{S}$ refining $\mathscr{P}$, with a subpresentation $\mathscr{Z}$ covering B, such that $\mathscr{S}$ regularly collapses* (n) *to* $\mathscr{Z}$.

Proof. There is a polyhedral equivalence $h : A \to I^n$, with $h(B) = T^n$. Then h is simplicial with reference to some $\mathscr{P}_1$ and Q, where $\mathscr{P}_{\alpha_n}$ can be assumed to refine Q. The diagram

$$I^n \xrightarrow{\alpha_n} I^{n-1} \to \ldots \xrightarrow{\alpha_2} I'$$

can be triangulated by simplicial presentations $\mathscr{S}_n, \ldots, \mathscr{S}_1$, where $\mathscr{S}_n$ can be assumed to refine Q. By 7.1.3, $\mathscr{S}_n$ regularly collapses (n) to $\mathscr{Z}_n$, the subpresentation of $\mathscr{S}_n$ covering T_n. Therefore the isomorphic presentation $h^{-1}(\mathscr{S}_n) = \mathscr{S}$ collapses regularly (n) to $h^{-1}(\mathscr{Z}_n) = \mathscr{Z}$. □

Suppose that $\mathscr{S}$ is a simplicial presentation of an n-cell A, regularly collapsing (n) to $\mathscr{Z}$, $(|\mathscr{Z}|) = B$, an $(n-1)$-cell in ∂A. Let the intermediate stages be

$$\mathscr{S} = \mathscr{S}_1, \ldots, \mathscr{S}_k = \mathscr{Z},$$

176 where is obtained from $\mathscr{S}_i$ by a regular collapse (n) at outer edge σ_i and major simplex $\Delta_i = \sigma_i \tau_i$.

We define the upper boundary of $\mathscr{S}_i$ as follows:

upper boundary of $\mathscr{S}_1 = \partial(|\mathscr{S}_1|)$-interior $(|\mathscr{Z}|)$

upper boundary of $\mathscr{S}_{i+1}$

= (upper boundary of $\mathscr{S}_i - \overline{\sigma}_i * \partial\tau_i) \cup \partial\sigma_i * \overline{\tau}_i$.

It can be alternatively defined as follows: Upper boundary of $\mathscr{S}_i$ = unions of closures of $(n-1)$-cells E of $\mathscr{S}_i$, such that if $E \notin \mathscr{Z}$, E is the face of exactly one n-simplex of $\mathscr{S}_i$ and if $E \in \mathscr{Z}$ then E is the face of no n-simplex of $\mathscr{S}_i$.

Now we would like to assert that

7.1.5

(a) The upper boundary of $\mathscr{S}_i$ is an $(n-1)$-cell, with constant boundary $\partial(|\mathscr{Z}|)$. The upper boundary of the last stage is $|\mathscr{Z}|$.

(b) $\overline{\Delta}_i$ intersects the upper boundary of precisely along $\overline{\sigma}_i * \partial\tau_i$. In particular τ_i cannot be in the upper boundary of $\mathscr{S}_i$ for any i, hence can never be in $\partial|\mathscr{Z}|$.

If in 7.1.3, in each column we do the collapsing as described in 7.1.2, the above assertions can be verified in a straight forward manner, by using similar properties of $\mathscr{S}_{n-1}$ and an analysis of the individual steps in 7.1.2. The general case seems to be more cumbersome (A proof is given in the appendix). But the special case is enough for our purposes, namely for the next theorem, the main result of this chapter.

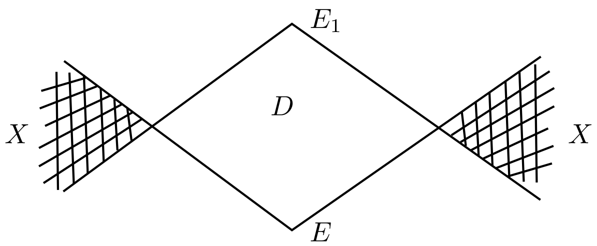
First using 7.1.5, we define a polyhedral equivalence φ_i from the
177 upper boundary of $\mathscr{S}_i$ to the upper boundary of $\mathscr{S}_{i+1}$ by φ_i = identity

outside $\overline{\sigma}_i * \partial\tau_i$, and on $\overline{\sigma}_i * \partial\tau_i$, it is the join of the identity map $\partial\sigma_i * \partial\tau_i$ to the map of centre of σ_i to the centre of τ_i.

Thus from $\overline{\partial|\mathscr{S}| - |\mathscr{Z}|}$ to $|\mathscr{Z}|$, we reach by simplicial moves, never disturbing the boundary of $|\mathscr{Z}|$.

Theorem 7.1.6. *Let D be a $(k+1)$-cell contained in the interior of an n-cell Δ. Let $\partial D = E_1 \cup E_2$, E_1 and E_2 two k-cells, $\partial E_1 = \partial E_2$; let $X \subset \Delta$ be a polyhedron such that $X \cap D \subset \partial E_1$. Then there is an isotopy of Δ, fixed on $X \cup \partial\Delta$, taking E_1 onto E_2.*

Proof.



Consider Δ to be a standard n-cell, we can suppose that $\partial\Delta \subset X$, and triangulate the whole picture, so that there are subpresentations covering D, X. Refine the subpresentation covering D, to $\mathscr{S}$, which regularly collapses $(K+1)$ to $\mathscr{Z}$ which covers E_2. Extend $\mathscr{S}$ to the whole of Δ, to say $\mathscr{P}$. Let the intermediate stages of the collapsing be

$$\mathscr{S} = \mathscr{S}_1, \ldots, \mathscr{S}_p = \mathscr{Z},$$

$\mathscr{S}_{i+1}$ obtained from $\mathscr{S}_i$ by an elementary regular collapse $(k+1)$ at out **178**
edge σ_i and major simplex $\Gamma_i = \sigma_i\tau_i$.

We will find an isotopy taking the upper boundary of $\mathscr{S}_i$ to the upper boundary of $\mathscr{S}_{i+1}$, and fixed except in a certain n-cell to be described.

Γ_i is a $(k+1)$-simplex contained in the complement of X, which is also covered by a subpresentation of $\mathscr{P}$. So if take $|\lambda_{\mathscr{P}}\Gamma_i| = \sum$, say, $\sum * \overline{\Gamma}_i \subset \Delta$, and $(\sum * \overline{\Gamma}_i) \cap X = \overline{\Gamma}_i \cap X$. Now $\overline{\Gamma}_i \cap X$ must be contained in $\partial\sigma_i * \partial\tau_i$, for this is the only part of $\overline{\Gamma}_i$ which could contain points in ∂E_i. Let s and t be the centres of σ_i and τ_i, the line segment $[s, t]$ can be prolonged a little bit (here we use the fact that Δ is standard) to v and w in Δ, so that

$$\Big([v,w]*(\partial\sigma_i*\partial\tau_i)*\sum\Big)\cap X\subset\partial\sigma_i*\partial\tau_i$$
$$\Big([v,w]*(\partial\sigma_i*\partial\tau_i)*\sum\Big)\cap\ \text{upper boundary}$$

of $\mathscr{S}_i\subset\partial\sigma_i*\partial\tau_i$. (here we use the fact that if $L\cap(\overline{\sigma}*K)\subset L\cap K$, where σ is a simplex and K, L are polyhedra, then there is a stretching σ' of σ i.e. containing $\overline{\sigma}$ such that $L\cap(\overline{\sigma}'*K)\subset K\cap L$). Thus we have in order $\{v,s,t,w\}$ and there is a polyhedral equivalence f of $[v,w]$, taking v to v, s to t and w to w. Join f to the identity on $\partial\sigma_i*\partial\tau_i*\sum$ and extend by identity outside of $[v,w]*\partial\sigma_i*\partial\tau_i*\sum$; call it h_i. Now h_i is the result of a nice isotopy and takes the upper boundary of $\mathscr{S}_i$ to the upper boundary of $\mathscr{S}_{i+1}$.

179 The composition of the h_i, will then take the upper boundary of $\mathscr{S}_1=E_1$ to the upper boundary of $\mathscr{S}_p=E_2$. □

Remark 7.1.7. In theorem 7.1.6, Δ can be replaced by any PL-manifold. Of course D should be in the interior.

Ex. 7.1.8. If N and M are two PL-manifolds and $f:N\times I\to\operatorname{int}M$ an imbedding, show that there is an isotopy of M fixing ∂M and carrying $f(N\times 0)$ to $f(N\times 1\cup\partial N\times I)$. If X is a polyhedron in M, and $X\cap f(N\times I)\subset\partial f(N\times 0)$, the isotopy can be chosen to leave X fixed.

7.2 Applications

Definition 7.2.1. Let S be an n-sphere, and $\sum$ a k-sphere in S. The pair $(S,\sum)$ is said to be *unknotted* if $(S,\sum)$ is polyhedrally equivalent to $(X*\sum,\sum)$ for some X.

X must of course be an $(n-k-1)$-sphere. Clearly a pair equivalent to an unknotted pair is again unknotted.

Proposition 7.2.2. *Let S be an n-sphere, and $\sum$ a k-sphere in S. If there exists an $(n-k-1)$-cell D in S such that $D*\sum\subset S$, then $(S,\sum)$ is unknotted.*

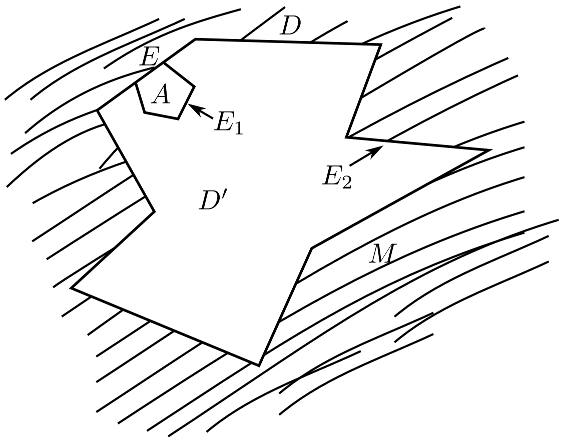
Proof. $D * \sum$ is an n-cell, and so the closure of $S - D * \sum$, say Δ, is again an n-cell and $\partial\Delta = \partial(D * \sum) = \partial D * \sum$. Then S is polyhedrally equivalent to a suspension of $\partial D * \sum$, hence $(S, \sum)$ is equivalent $(X * \sum, \sum)$ where X is a suspension of ∂D. □

Corollary 7.2.3. *If $\mathscr{P}$ is a regular presentation of an n-sphere S, and A a $(k+1)$-cell in $\mathscr{P}$, then $(S, \partial A)$ is unknotted.*

Proof. Take $D = |\delta_{\mathscr{P}} A|$ (with respect to some centering of $\mathscr{P}$) in 7.2.2. **180**
□

Proposition 7.2.4. *If a k-sphere $\sum$ bounds a $(k+1)$-cell D contained in the interior of a PL-manifold M, then there is an isotopy of M taking $\sum$ onto the boundary of a $(k+1)$-cell of some regular presentation of M.*

Proof. Take a regular presentation $\mathscr{P}$ of M in which D is covered by a full subpresentation Q. Consider a k-cell E of Q in ∂D and the $(k+1)$-cell, say A of Q, which contains it in its boundary. Let $\overline{\partial A - E} = E_1$ and $\overline{\partial D - E} = E_2$ and $\overline{D - A} = D'$. Then D' is a $(k+1)$-cell with boundary $E_1 \cup E_2$ and E intersect D' in $\partial E_1 = \partial E_2$. Hence by theorem 7.1.6, there is an isotopy of M taking E_2 onto E_1 and fixing E. Thus ∂D will be moved onto ∂A.



□

Corollary 7.2.5. *Let S be an n-sphere, and $\sum$ a k-sphere in S. $(S, \sum)$ is unknotted if and only if $\sum$ bounds a $(k+1)$-cell in S.*

Proof. The necessity is clear. Sufficiency follows from 7.2.4 and 7.2.3. ☐

Motivated by 7.2.4, we define a k-sphere $\sum$ in the *interior* of a PL-
181 manifold M to be *unknotted* if it bounds a $(k+1)$-cell in (the interior of) M. From 7.2.4 it is clear that

7.2.6 If A is a $(k+1)$-cell of some regular presentation of M, and $\overline{A} \subset \operatorname{int} M$, then ∂A is unknotted. If $\sum_1$ and $\sum_2$ are two unknotted spheres in the same component of M, there is an isotopy of M which takes $\sum_1$ onto $\sum_2$ keeping M fixed.

Definition 7.2.7. If D is an n-cell and E a k-cell in D, with $\partial D \subset \partial E$, (D, E) is said to be *unknotted if* (D, E) is polyhedrally equivalent to $(X * E, E)$ for some X.

Since E is not completely contained in ∂D, such an X must be an $(n-k-1)$-sphere.

And we define a cell E in the *interior* of a PL n-manifold M to be *unknotted*, if there is an n-cell D in M containing E such that (D, E) is unknotted. A cell which is the closure of an open convex cell of some regular presentation of M is clearly unknotted. Given any two unknotted cells D_1 and D_2 of the same dimension in M, there is an isotopy of M leaving ∂M fixed and taking D_1 onto D_2. Given two unknotted k-cells D_1 and D_2 in a PL n-manifold M, $k < n$, $D_1 \cap D_2 = \emptyset$, then there is a n-cell A containing D_1 and D_2 in D and such that the triple (A, D_1, D_2) is equivalent to a standard triple. In particular if $k \leq n-2$, from the standard situation, we see that there is a $(k+1)$-cell A in int M containing D_1 and D_2 in ∂A and inducing chosen orientations on D_1 and D_2. These remarks will be used in the next chapter.

182 Now, as a corollary of 7.2.5, if $\sum^k \subset S^n$ are k and n-spheres and $n \geq 2k+2$, then $(S^n, \sum^k)$ is unknotted. The next case $n = 2k+1$ is a little more difficult. Actually $n - k \geq 3$ is enough. But this will be proved only in the next chapter. Here we sketch a proof of the case $n = 2k+1$.

Proposition 7.2.8. *Let S be an n-sphere, $\sum$ a k-sphere in S, $n = 2k + 1$, and $k2$. $(S, \sum)$ is unknotted.*

Sketch of the proof: By 7.2.5 it is enough to show that $\sum$ bounds a $(k + 1)$-cell in S. To prove this it is enough to show that a k-sphere in $\mathbb{R}^{2k+1}$ bounds a $(k + 1)$-cell. Consider a k-sphere P in $\mathbb{R}^{2k+1}$ and let $\mathscr{P}$ be a simplicial presentation of P. If σ and τ are two $(\leq k)$-dimensional simplexes in $\mathbb{R}^{2k+1}$ and L_σ and L_τ the linear manifolds generated by them, $\sigma\tau$ is defined if and only if given any point $x \in \mathbb{R}^{2k+1}$, there is at most one line through x meeting L_σ and L_τ.

Consider $L = \cup\{L_{(\sigma,\tau)} | L_{(\sigma,\tau)}$ the linear manifold generated by σ, $\tau \in \mathscr{P}$, for which $\sigma\tau$ is *not* defined.$\}$

The dimension of all such $L_{(\sigma,\tau)} \leq 2k$, hence $\mathscr{U} = \mathbb{R}^{2k+1} - L$ is open and dense in $\mathbb{R}^{2k+1}$. By the above remark, if we take any point $x \in \mathscr{U}$, then for any (σ, τ), $\sigma \in \mathscr{P}$, $\tau \in \mathscr{P}$, at most one line through x meets σ and τ, that is, at most a finite number of lines through x meet P more than once. But each of these finite number lines through x may meet P more than twice. By similar arguments using triples (σ, τ, ρ), σ, τ, $\rho \in \mathscr{P}$, we can get an open dense set $\mathscr{U}' \subset \mathbb{R}^{2k+1}$ such that if $x \in \mathscr{U}'$, only a finite number of lines meet P more than once, and each such meets P exactly twice. Now we choose such a point x; let $L_1, \ldots, L_p$ **183**
be the lines through x which meet P at two points. On each L_i, call the point on P nearer to x as N_i, and the other F_i, and consider the set $N_1, \ldots, N_p$. If $k \geq 2$, we can put $N_1, \ldots, N_p$ is a 1-cell in P not meeting F_i. Let N be a regular neighbourhood of that 1-cell in P. We can choose N so that $F_i \notin N$ for all i. N is a k-cell and (its complement in P) say F is another k-cell $x * N$ is a $(k + 1)$-cell, $\partial(x * N) = N \cup x * \partial N$, and F meets $x * N$, exactly in ∂N. Hence by theorem 7.1.6, here is an isotopy of $\mathbb{R}^{2k+1}$ taking N onto $x * \partial N$ and keeping F fixed. But now $(x * \partial) \cup F$ is the boundary of the $(k+1)$-cell $x * F$. Since P is moved to $(x * \partial N) \cup F$ by an isotopy, P also bounds some $(k + 1)$-cell.

Appendix to Chapter VII

184 In the theory of regular collapsing, let us add the following operation (due to J.H.C. Whitehead): also namely the operation of removing a principal simplex (open) from a simplicial presentation. This is called "perforation". If $\mathscr{S}$ is a simplicial presentation, and $\mathscr{S}'$ is obtained from by removing a principal i-simplex, we will say that "$\mathscr{S}'$ is obtained from $\mathscr{S}$ by a perforation of dimension i", or more briefly "$\mathscr{S}'$ is obtained from $\mathscr{S}$ by perforation (i)". If n the definition or regular collapsing, we did not put the restriction that the dimension of the major simplex should be greater than that of the outer edge, then perforation also would come under regular collapsing. Since regular collapsing as defined in 7.1 does not change the homotopy type (even the simple homotopy type), where as perforation does, we prefer to distinguish them.

A.1. Let $\mathscr{S}' \subset \mathscr{S}$ be simplicial presentations such that $\mathscr{S}'$ is contained from $\mathscr{S}$ by an elementary regular collapse (n) at outer edge σ and major simplex $\Delta = \sigma\tau$. Let $\rho \in \mathscr{S}'$. Then

(a) $Lk(\rho, \mathscr{S}) = Lk(\rho, \mathscr{S}')$ if ρ is not a face of Δ.

(b) If $\tau \leq \rho < \Delta$, then $Lk(\rho, \mathscr{S}')$ is obtained from $Lk(\rho, \mathscr{S})$ by a perforation of dimension $(n - \dim\rho - 1)$.

(c) If $\rho < \Delta$ and $\tau \nleq \rho$, then $Lk(\rho, \mathscr{S}')$ is obtained from $Lk(\rho, \mathscr{S})$ by an elementary regular collapse of dimension $(n - \dim\rho - 1)$.

The verification is easy. The only faces of Δ which are not covered
185 by (b) and (c) above are of those in $\mathscr{S} - \mathscr{S}'$, that is those which contain σ as face. Of course these do not appear in $\mathscr{S}'$.

Suppose $\mathscr{S}$ collapses regularly (n) to $\mathscr{Z}$. If $\rho \in \mathscr{S} - \mathscr{Z}$, ρ has to disappear in some collapse; let us denote the major simplex of the regular collapse (n) in which ρ is removed by Δ_ρ. If σ_ρ is the outer edge of the particular collapse, then $\sigma_\rho \leq \rho$. What all is left of $Lk(\rho, \mathscr{S})$ at this stage is $Lk(\rho\{\overline{\Delta}_\rho\})$. With this notation, using A.1, we have easily the following:

A.2

(a) If $\rho \in \mathscr{X}$, then $Lk(\rho, \mathscr{X})$ is obtained from $Lk(\rho, \mathscr{S})$ by perforations and regular collapses of dimension $(n - \dim \rho - 1)$.

(b) If $\rho \in \mathscr{S} - \mathscr{X}$, then $Lk(\rho, \{\overline{\Delta}_\rho\})$ is obtained from $Lk(\rho, \mathscr{S})$ by perforations and regular collapses of dimension $(n - \dim \rho - 1)$.

Let $\mathscr{B} \subset \mathfrak{a}$ be simplicial presentations and suppose $\mathscr{B}$ is obtained from $\mathfrak{a}$ by regular collapses and perforations of dimension i. Then we can rearrange the operations so that perforations come first and regular collapses later. This is easily seen by considering one perforation and one regular collapse. If the perforation comes after the regular collapse, we can reverse the order; of course the converse is not true. By a finite number of such changes, we can perform the perforations first and the regular collapses later, so that the end result is still $\mathscr{B}$. If $|\mathfrak{a}|$ is a connected PL(i)-manifold, the effect of a perforation (i) upto homotopy type is the same as removing a point from the interior of $|\mathfrak{a}|$. Since a regular collapse does not change the homotopy type, we have

A.3. If $|\mathfrak{a}|$ is a connected i-manifold, and $\mathscr{B}$ is obtained from by k perfo- **186**
rations (i) and certain elementary regular collapses (i), then $|\mathscr{B}|$ has the same homotopy type as $|\mathfrak{a}|$ with k interior points removed. In particular if $|\mathfrak{a}|$ is a i-cell then $|\mathscr{B}|$ has the homotopy type of a wedge of k spheres of dimension $(i - 1)$. If $|\mathfrak{a}|$ is a i-sphere then $|\mathscr{B}|$ has the homotopy type of a wedge of $(k - 1)$ spheres of dimension $(i - 1)$.

Of course, in the above when $i = 1$, the wedge of 0-spheres has to interpreted properly. That is we should take the wedge of k 0-spheres to be $(k+1)$ distinct points, in particular if $k = 0$ to be just a point. Suppose $|\mathfrak{a}|$ is a i-cell, and $|\mathscr{B}|$ has the homotopy type as point, for example when $|\mathscr{B}|$ is a i-cell or an $(i - 1)$-cell. Then there *cannot* be any perforations. If $|\mathfrak{a}|$ is a cell and $|\mathscr{B}| = \partial|\mathfrak{a}|$, there is exactly one perforation. If $|\mathfrak{a}|$ is a i-sphere and $|\mathscr{B}|$ and i-cell in it, again, there is exactly one perforation.

It should be remarked, that all the above statemetns are made for the sake of proving Lemma 7.1.5 to which we proceed now. Let us first recall the definition of the upper boundary. Consider $\mathscr{S}$, a simplicial presentation of an n-cell A regularly collapsing (n) to $\mathscr{X}$, where $|\mathscr{X}| = B$ is an $(n - 1)$-cell in A. Let the individual stages be

$$\mathscr{S} = \mathscr{S}_1, \ldots, \mathscr{S}_p = \mathscr{X},$$

where $\mathscr{S}_{i+1}$ is obtained from $\mathscr{S}_i$ by an elementary regular collapse (n) at outer edge σ_i and major simplex $\Delta_i = \sigma_i\tau_i$. (This is the hypothesis for the rest of the appendix). Then the upper boundary of $\mathscr{S}_i$ (denoted by $\overline{\sigma}(\mathscr{S}_i|\mathscr{Z})$) is defined inductively as follows:

$$\overline{\partial}(\mathscr{S}_1|\mathscr{Z}) = \overline{\partial A - B} = (\overline{\partial|\mathscr{S}_1| - |\mathscr{Z}|})$$
$$\overline{\partial}(\mathscr{S}_{i+1}|\mathscr{Z}) = \{\overline{\partial}(\mathscr{S}_i|\mathscr{Z}) - \overline{\sigma}_i * \partial\tau_i\} \cup \partial\sigma_i * \overline{\tau}_i.$$

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The trouble with this definition is that it is not clear that it is well defined, e.g. that $\overline{\sigma}_i * \partial\tau_i \subset \partial(\overline{\mathscr{S}}_i|\mathscr{Z})$. So we considr the following:

$\overline{\sigma}'(\mathscr{S}_i|\mathscr{Z}) = \cup\{\overline{E}|E$ is an $(n-1)$-simplex of $\mathscr{S}_i$ such that (1) if $E \in \mathscr{S}_i - \mathscr{Z}$ then E is the face of exactly one n-simplex of $\mathscr{S}_i$ (2) if $E \in \mathscr{Z}$ then E is the face of no n-simplex of $\mathscr{S}_i$.$\}$ We claim that $\overline{\partial}(\mathscr{S}_i|\mathscr{Z}) = \overline{\partial}'(\mathscr{S}_i|\mathscr{Z})$. To begin with they are equal, that is when $i = 1$. Suppose they are equal for i. Then we will show that they are equal for $i+1$ also. In $\overline{\partial}^1(\mathscr{S}_i|\mathscr{Z})$ and $\overline{\partial}^1(\mathscr{S}_{i+1}|\mathscr{Z})$, the only changes can be from faces of Δ_i. Now all the $(n-1)$-simplexes in $\{\overline{\sigma}_i\} * \{\partial\tau_i\}$ have to be in $\delta^1(\mathscr{S}_i|\mathscr{Z})$ since Δ_i is the only n-simplex of $\mathscr{S}_i$ having them as faces. So by induction $\overline{\sigma}_i * \partial\tau_i$ is really in $\overline{\partial}(\mathscr{S}_i|\mathscr{Z})$. Now consider $\overline{\partial}^1(\mathscr{S}_{i+1}|\mathscr{Z})$. None of the $(n-1)$-simplexes of $\{\overline{\sigma}_i\} * \{\partial\tau_i\}$ is in this, since they are not in $\mathscr{S}_{i+1}$. The $(n-1)$-simplexes of $\{\partial\sigma_i\} * \{\overline{\tau}_i\}$ have to be in $\overline{\sigma}^1(\mathscr{S}_{i+1}|\mathscr{Z})$. For, consider any $(n-1)$-simplex E of $\{\partial\sigma_i\} * \{\overline{\tau}_i\}$. If E is in $\mathscr{Z}$, then Δ_i is the only n-simplex of $\mathscr{S}$ having E as face, since that is removed there is no n-simplex of $\mathscr{S}_{i+1}$ having E as a face. If $E \in \mathscr{S} - \mathscr{Z}$, there are two n-simplexes in $\mathscr{S}$ having E as a face. One of them Δ_i is removed.
188 The other should be in $\mathscr{S}_{i+1}$, since otherwise E cannot be removed in any of the later collapses. Thus $\partial\sigma_i * \tau_i$ is in $\overline{\partial}^1(\mathscr{S}_{i+1}|\mathscr{Z})$. Since we have accounted for all the $(n-1)$-faces of Δ_i, these are the only changes from $\overline{\partial}'(\mathscr{S}_i|\mathscr{Z})$ to $\overline{\partial}'(\mathscr{S}_{i+1}|\mathscr{Z})$, that is

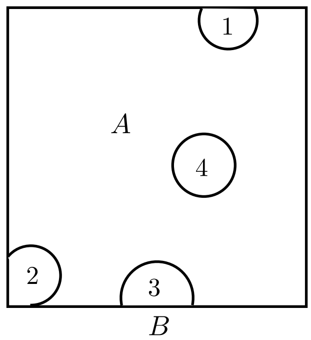
$$\overline{\partial}^1(\mathscr{S}_{i+1}|\mathscr{Z}) = \{\overline{\partial}^1(\mathscr{S}_i|\mathscr{Z}) - \overline{\sigma}_i * \partial\tau_i\} \cup (\partial\sigma_i * \overline{\tau}_i).$$

Hence by induction $\overline{\partial}$ and $\overline{\partial}^1$ coincide for all i, and $\overline{\partial}$ is well defined.

A.4. $\tau_i \not\subset \overline{\partial}(\mathscr{S}_i|\mathscr{Z})$.

Proof. Suppose $\tau_i \subset \overline{\partial}(\mathscr{S}_i|\mathscr{Z})$. There are gour possibilities:

$$
\begin{array}{lll}
\text{Either} & (1) & \tau_i \subset \partial A - B \\
\text{or} & (2) & \tau_i \subset \partial B \\
\text{or} & (3) & \tau_i \subset B - \partial B \\
\text{or} & (4) & \tau_i \subset A - \partial A.
\end{array}
$$



We will show that $\tau_i \subset \overline{\partial}(\mathscr{S}_i|\mathscr{Z})$ is impossible in each case.

By A.2 in cases (2) and (3) $Lk(\tau_i, \mathscr{Z})$ is obtained from $Lk(\tau_i, \mathscr{S})$ by perforations and regular collapses of dimension $(n - \dim \tau_i - 1)$. In cases (1) and (4) $Lk(\tau_i, \{\overline{\Delta}_{\tau_i}\})$ (with the notation of A.2) is obtained from $Lk(\tau_i, \mathscr{S})$ by perforations and regular collapses of dimension $(n - \dim \tau_i - 1)$. By A.3 and remarks thereafter, there cannot be any perforations in cases (1) and (2) and there is exactly one perforation in cases (3) and (4).

By A.1; (b) in the collapse at outer edge σ_i and major simplex $\Delta_i = \sigma_i\tau_i$, what happens to $Lk(\tau_i, \mathscr{S}_i)$ is exactly a perforation of dimension **189**
$(n - \dim \tau_i - 1)$. So straightaway we have $\tau_i \subset \partial A - B$ or $\tau_i \subset \partial B$ is impossible.

So, the only possibilities that remain are (3) and (4). Let us consider case (4) first. We claim that if $\tau_i \subset \overline{\partial}(\mathscr{S}_i|\mathscr{Z})$ the one perforation on $Lk(\tau_i, \mathscr{S})$ is already made. Since $|Lk(\tau_i, \mathscr{S})|$ is a sphere, any $(n-1)$-simplex of $\mathscr{S}$ having τ_i as a face must be the face of two n-simplexes. So τ_i cannot be in $\overline{\partial}(\mathscr{S}_1|\mathscr{Z})(\mathscr{S} = \mathscr{S}_1)$. For the same reason, τ_i cannot be in any $\overline{\partial}(\mathscr{S}_j|\mathscr{Z})$ with $Lk(\tau_i, \mathscr{S}_j) = Lk(\tau_i, \mathscr{S}_1)$. Thus $\tau_i \subset \overline{\partial}(\mathscr{S}_i|\mathscr{Z})$

implies $Lk(\tau_i, \mathscr{S}_i) \neq Lk(\tau_i, \mathscr{S}_1)$. Suppose $Lk(\tau_i, \mathscr{S})$ is changed for the first time in the k_i^{th} collapse, $k_i < i$, that is $Lk(\tau_i, \mathscr{S}_{k_i}) = Lk(\tau_i, \mathscr{S}_1)$, but $Lk(\tau_i, \mathscr{S}_{k_i+1}) \neq Lk(\tau_i, \mathscr{S}_1)$. Since $Lk(\tau_i, \mathscr{S}_1)$ is a sphere; this operation from $Lk(\tau_i, \mathscr{S}_1) = Lk(\tau_i, \mathscr{S}_{k_i})$ to $Lk(\tau_i, \mathscr{S}_{k_i+1})$ is necessarily a perforation. So the one perforation on $Lk(\tau_i, \mathscr{S})$ is already made. But in the i^{th} collapse also what happens to $Lk(\tau_i, \mathscr{S}_i)$ is a perforation since $\Delta_i = \sigma_i \tau_i$ (by A.1.b). Since this is impossible τ_i cannot be in $A - \partial A$.

Let us consider the remaining possibility (3), $\tau_i \subset B - \partial B$. $|Lk(\tau_i, \mathscr{S})|$ is an i-cell with boundary $|Lk(\tau_i, \mathscr{Z})|$. If $\tau_i \subset \overline{\partial}(\mathscr{S}_i|\mathscr{Z})$, we have to show that $\tau_i \subset B - \partial B$ is also impossible. The case when $\dim \tau_i = n - 1$ is easily disposed of, since in that case there is no n-simplex having τ_i as a face. As in case (4) τ_i is not in $\overline{\partial}(\mathscr{S}_1|\mathscr{Z})$ and τ_i cannot be in $\overline{\partial}(\mathscr{S}_j|\mathscr{Z})$ if
190 $Lk(\tau_i, \mathscr{S}_1) = Lk(\tau_i, \mathscr{S}_j)$. Again, the first operation on $Lk(\tau_i, \mathscr{S}_1)$ has to be a perforation. For, all the outer edges of $Lk(\tau_i, \mathscr{S}_1)$ are in $Lk(\tau_i, \mathscr{Z})$, and a regular collapse of $Lk(\tau_i, \mathscr{S}_1)$ removes a part of $Lk(\tau_i, \mathscr{Z})$. Thus $\tau_i \subset \overline{\partial}(\mathscr{S}_i|\mathscr{Z})$ implies that the one perforation on $Lk(\tau_i, \mathscr{S}_1)$ is already done. But then the result of the i^{th} collapse will be again a perforation on $Lk(\tau_i, \mathscr{S}_i)$ by A.1.b) since $\Delta_i = \sigma_i \tau_i$. So this is again impossible.

Thus τ_i cannot be in $\overline{\partial}(\mathscr{S}_i|\mathscr{Z})$ for any i. □

A.5. With the hypothesis of A.4., $\overline{\partial}(\mathscr{S}_i|\mathscr{Z})$ is an $(n-1)$-cell with constant boundary $= \partial(|\mathscr{Z}|) = \partial B$.

Proof. $\overline{\partial}(\mathscr{S}_1|\mathscr{Z})$ is an $(n-1)$-cell with boundary $= \partial B$. Inductively, assume that $\overline{\partial}(\mathscr{S}_i|\mathscr{Z})$ is an $(n-1)$-cell with boundary ∂B. By A.4, $\tau_i \not\subset \overline{\partial}(\mathscr{S}_i|\mathscr{Z})$; in particular it cannot be in ∂B. Since $\tau_i \not\subset \overline{\partial}(\mathscr{S}_i|\mathscr{Z})$, no simplex of $\mathscr{S}$ having τ_i as a face can be in $\overline{\partial}(\mathscr{S}_i|\mathscr{Z})$. So $\partial\sigma_i * \overline{\tau}_i$ intersects $\overline{\partial}(\mathscr{S}_i|\mathscr{Z})$ precisely along $\partial\sigma_i * \partial\tau_i$. Define $\partial_i : \overline{\partial}(\mathscr{S}_i|\mathscr{Z}) \to \overline{\partial}(\mathscr{S}_{i+1}|\mathscr{Z})$ by φ_i: Identity outside $\overline{\sigma}_i * \partial\tau_i$, and on $\overline{\sigma}_i * \partial\tau_i$, φ_i is the join of the identity map on $\partial\sigma_i * \partial\tau_i$ and the map which carries the centre of σ_i to the centre of $\tau_i \cdot \varphi_i$ is clearly a polyhedral equivalence; hence $\overline{\partial}(\mathscr{S}_{i+1}|\mathscr{Z})$ is an $(n-1)$-cell. To see that $\partial(\overline{\partial}(\mathscr{S}_i|\mathscr{Z})) = \partial(\overline{\partial}(\mathscr{S}_{i+1}|\mathscr{Z}))$, observe that the part of $\overline{\sigma}_i * \partial\tau_i$ (if any) which is in $\partial(\overline{\partial}(\mathscr{S}_i|\mathscr{Z}))$ should be in $\partial\sigma_i * \partial\tau_i$. Since φ_i is identity on this part, both the cells have the same boundaries. □

Chapter 8

Handles and s-cobordism

8.1 Handles

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A *handle of dimension n* and *index k*, briefly called a (n, k)-handle, (or a k-handle) is a pair (H, T) consisting of an n-cell H and $(n-1)$-manifold T of ∂H, such that there is a polyhedral equivalence

$$f : H \approx A * B$$

where A is a $(k-1)$-sphere, B a $(n-k-1)$-sphere, and $f(T)$ a regular neighbourhood of A in $A * B$.

We denote handles by lower case script letters, as $\mathfrak{h}$, $\mathfrak{K}$, and so on.

Given a handle (H, T) as above, we call T the *attaching tube* and $\overline{\partial H - T}$ the *transverse tube* of the handle. The polyhedral equivalence f in the definition can be so that $f(T) = \varphi^{-1}([0, \frac{1}{2}])$, where $\varphi : A*B \to 0, 1$ is the join of $A \to 0$ and $B \to 1$. When this is so, $f^{-1}(A)$ is called an *attaching sphere* and $f^{-1}(B)$ a *transverse sphere* of the handle.

The pair $(H, \overline{\partial H - T})$ is clearly a handle of dimension n and index $n - k$. It is called the *dual* of (H, T), and denoted by $(H, T)^*$.

The cone on X is denoted by $C(X)$. We know that, by a standard mistake, $C(A*B) \approx C(A) \times C(B)$. This equivalence will make $\varphi^{-1}([0, \frac{1}{2}])$ correspond to $A \times (C(B))$. Therefore, in defining a handle, we could require, in place of f, the existence of a polyhedral equivalence 192

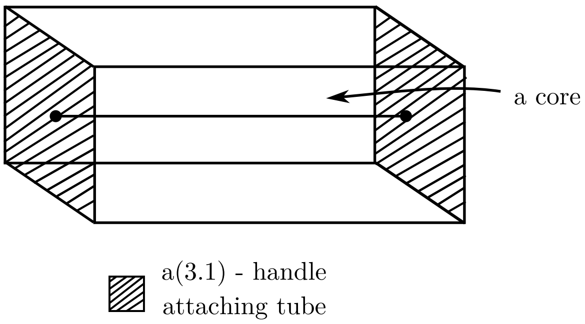
$$g : H \approx D \times \Delta$$

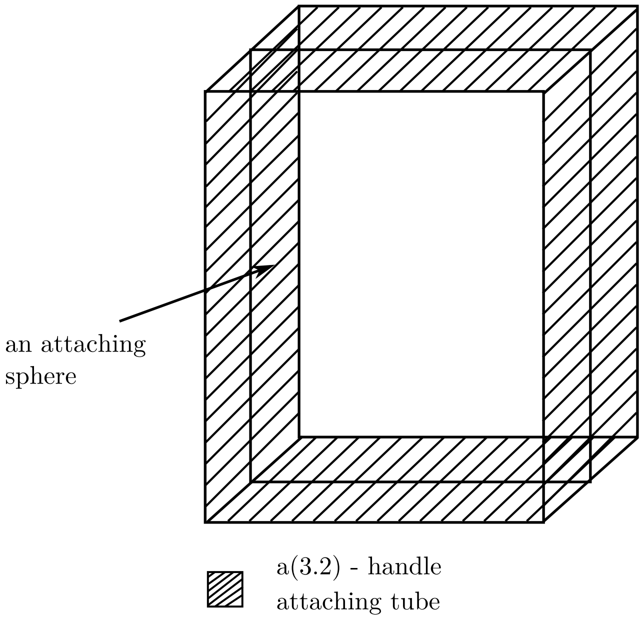
where D is a k-cell, Δ an $(n-k)$-cell, and where $g(T) = (\partial D) \times \Delta$.

With this formulation, for any e in the interior of Δ, then $\partial D \times e$ is an attaching sphere; and for any f in the interior of Δ, then $f \times \partial\Delta$ is a transverse sphere, in the handle $(D \times \Delta, (\partial D) \times \Delta)$. If $e \in \operatorname{int} \Delta$, we call $D \times e$ a *core* of the handle. If $e \in \partial\Delta$, we call $D \times e$ a *boundary core* or a *surface core* of the handle. Similarly transverse cores are defined, and the definitions can be extended to arbitrary handles by using an equivalence with the standard handle (so that even in the standard handle, we have "more" cores than defined above). Note that there is no uniqueness about attaching spheres, transverse spheres and cores in a handle, only the attaching tube and the transverse tube are fixed.

Ex. 8.1.1. If H is an n-cell, and S a $(k-1)$-sphere in ∂H, S is an attaching sphere of some (n, k)-handle (H, T) if and only if S is unknotted in ∂H.

We have the following two extreme cases of (n, k)-handles: If (H, T) is a $(n, 0)$-handle there is no attaching sphere ($T = \emptyset$), ∂H is the transverse tube as well as *the* transverse sphere. Any point in the interior of H can be considered as a core. If (H, T) is a (n, n)-handle, H is the attaching tube as well as the attaching sphere, the whole of H is the core. Also, note that for an $(n, 1)$-handle, the attaching tube consists of two disjoint $(n-1)$-cells.





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8.2 Relative n-manifolds and their handle presentations

A relative n-manifold is a pair (M, X), $X \subset M$, such that for every $a \in M - X$, the link of a in M is either an $(n-1)$-cell or an $(n-1)$-sphere. If

(M, X) is a relative n-manifold, $\partial(M, X)$ denotes the set of points of $M - X$ whose links are cells. $\partial(M, X)$ is not a polyhedron, but $\partial(M, X) \cup X$ and $\overline{\partial(M, X)} = \partial(M, X) \cup (X \cap \overline{\partial(M, X)})$ are polyhedra; so that $(\partial(M, X) \cup X, X)$ and $(\overline{\partial(M, X)}, X')$ (where $X^1 = X \cap \overline{\partial(M, X)}$) are relative $(n-1)$-manifolds without boundary. Any compact set in $\partial(M, X)$ is contained in an $(n-1)$-manifold contained in $\partial(M, X)$.

194 We sometimes denote a relative manifold (M, X) by Gothic letter such as $\mathscr{M}$, and $\partial(M, X)$ by $\partial\mathscr{M}$.

If (M, X) is a relative n-manifold, and A an n-manifold, such that $A \cap M = \partial A \cap \partial(M, X)$ is an $(n-1)$-manifold, then it is easily proved that (using, of course, theorems on cells in spheres etc..) that $(M \cup A, X)$ is a relative n-manifold. As in the case of the manifolds, we have the following proposition:

Proposition 8.2.1. *Let (M, X) be a relative n-manifold, C an n-cell such that $C \cap M = \partial C \cap \partial(M, X)$ is an $(n-1)$-cell. Let $\mathscr{U}$ be any neighbourhood of $C \cap M$ in M. Then there is an equivalence*

$$f : (M, X) \approx (M \cup C, X)$$

which is identity outside $\mathscr{U}$.

Let $B \subset \partial A$, and $f : B \to M$ be an embedding with $f(B) \subset \partial(M, X)$, and B an $(n-1)$-manifold. Then there is an identification polyhedron $M \cup_f A$; and with the obvious convention of not distinguishing notationally between X and its image in $(M \cup_f A)$, we have $(M \cup_f A, X)$ is a relative n-manifold, which we shall say is obtained from (M, X) by *attaching (A, B) by an embedding f*. Of course, doing all this rigorously involves abstract simplicial complexes, their realizations and proper abuse of notation; and we assume that this is done in each case without mention.

Let $\mathscr{M} = (M, X)$ be a relative n-manifold, and $\mathfrak{h}_1, \ldots, \mathfrak{h}_p$ be (n, i)-handles, $\mathfrak{h}_j = (H_j, T_j)$. We speak of $\mathscr{M} + \mathfrak{h}_1 + \cdots + \mathfrak{h}_p$, when

(1) $H_i \cap H_j = \emptyset$

195 (2) $H_i \cap M = T_i \subset \partial\mathscr{M}$, for all i.

In such a case by definition,

$$\mathscr{M} + \mathfrak{h}_1 + \cdots + \mathfrak{h}_p = (M \cup H_1 \cup \ldots \cup H_p, X).$$

And we say that $\mathscr{M} + \mathfrak{h}_1 + \cdots + \mathfrak{h}_p$ is obtained from $\mathscr{M}$ by attaching $p(n, i)$-handles or p i-handles.

Also if we have $f_i : T_i \to \partial\mathscr{M}$ embeddings for $i = 1, \ldots, p$ and $f_i(T_i) \cap f_j(T_j) = \emptyset$ for $i \neq j$, we may look at what we obtain from $\mathscr{M}$ by attaching $\mathfrak{h}_1, \ldots, \mathfrak{h}_p$ by the maps $f_1, \ldots, f_p$. The result we denote by $\mathscr{M} \cup_{f_1} \mathfrak{h}_1 \cup \ldots \cup_{f_p} \mathfrak{h}_p$ and say that it is obtained from $\mathscr{M}$ by attaching $p(n, i)$-handles by imbeddings f_i.

Definition 8.2.2. A *handle presentation* of a relative n-manifold (M, X) is a $(n + 2)$-tuple $\mathscr{H} = (A_{-1}, \ldots, A_n)$, of polyhedra such that,

(1) $X \subset A_{-1} \subset \ldots \subset A_n = M$

(2) $A_{-1} \searrow X$

(3) $(A_i, X) = \mathfrak{a}_i$ is a relative n-manifold for all i

(4) For each i, there exist finitely many handles of index i $\mathfrak{h}_1^{(i)}, \ldots, \mathfrak{h}_{p_i}^{(i)}$, such that $\mathfrak{a}_i = \mathfrak{a}_{i-1} + \mathfrak{h}_1^{(i)} + \cdots + \mathfrak{h}_{p_i}^{(i)}$.

If follows from (3) and (4) that A_{-1} is a neighbourhood of X in M. $A_{-1} \searrow X$ implies that $A_{-1} \searrow N$, for some regular neighbourhood N of X in M (see Chapter 6). We can even assume that $N \subset \operatorname{int}_M A_{-1}$. Now if $B = b\mathfrak{d}_M N$, then $\overline{A_{-1} - N} \searrow B$, hence is a collar over B. Thus there is **196**
an equivalence of A_{-1} to N which fixex X; that is polyhedrally A_{-1} just looks like a regular neighbourhood of X in M.

Consider a relative n-manifold (M, X) where M is a PL n-manifold, and X a PL $(n-1)$-manifold in M. Such a relative manifold, we term a *special case*. If (M, X) is a special case, and $\mathscr{H} = (A_{-1}, \ldots, A_n)$ is a handle presentation of (M, X), then clearly $A_{-1} \approx X \times L$, moreover the equivalence can be assumed to carry x to $(x, 0)$ for $x \in X$.

Theorem 8.2.3. *Every relative manifold has a handle presentation.*

Proof. Let $\mathscr{M} = (M, X)$ be a relative n-manifold; let $\mathscr{P}$ be a regular presentation of M with a subpresentation $\mathscr{X}$ covering X. With a centering η of $\mathscr{P}$, we define the derived subdivision $d(\mathscr{P}, \eta)$, and some derived subdivision $d^2\mathscr{P}$ of $d(\mathscr{P}, \eta)$. Define

$$\begin{aligned} C^* &= |St(\eta C, d^2\mathscr{P})|, \quad \text{for} \quad C \in \mathscr{P} \\ A_{-1} &= \cup\{C^* | C \in \mathscr{X}\} \\ A_k &= \cup\{C^* | C \in \mathscr{X}, \quad \text{or} \quad C \in \mathscr{P} \quad \text{and} \quad \dim C \leq k\}. \end{aligned}$$

To thow that $\mathscr{H} = (A_{-1}, \ldots, A_n)$ is a handle presentation of $\mathscr{M}$, we note:

(1) $A_{-1} = |N_{d(\mathscr{P},\eta)}(d\mathscr{X})|$ is a regular neighbourhood of X in M.

(2) (A_{-1}, X) is a relative manifold. In fact, if Q is any subset of $\mathscr{P}$, containing $\mathscr{X}$, and Q^* denotes $\cup\{C^* | C \in Q\}$, then (Q^*, X) is a relative n-manifold.

197 These are easily proved.

The only thing that remains to be shown is that $A_{-k} = A_{k-1} + k$-handles. The k-handles evidently have to be $(C^*, C^* \cap \{\partial C\}^*)$, for $C \in \mathscr{P} - \mathscr{X}$ and $\dim C = k$. There are two different cases to consider, depending on whether C is in the interior or boundary of $\mathscr{M}$. Any how, C^* is an n-cell, since $\eta C \in M - X$ and (M, X) is a relative n-manifold.

There is a canonical isomorphism

$$Lk(\eta C, d^2\mathscr{P}) \approx d(Lk(\eta C, d\mathscr{P})),$$

which for $D < C$, takes $C^* \cap D^*$ to

$$D^+ = |St(\eta D, d(Lk(\eta C, d\mathscr{P})))|.$$

This shows that $C^* \cap \{\partial C\}^*$ corresponds to

$$N_{Lk(\eta C, d\mathscr{P})}(d\{pC\}) \quad \text{in} \quad d(Lk(\eta C, d\mathscr{P})).$$

A further fact is:

$$Lk(\eta C, d\mathscr{P}) = d\{\partial C\} * \lambda C.$$

Now if C is an interior k-cell, $|\lambda C|$ is an $(n-k-1)$-sphere; and so, composing all these facts together, we get a polyhedral equivalence f : $\partial(C^*) \approx \partial C * |\lambda C|$ which takes $C^* \cap \{\partial C\}^*$ onto a regular neighbourhood of ∂C. This directly shows that $(C^*, C^* \cap \{\partial C\}^*)$ is a k-handle.

If C is a boundary k-cell, then $|\lambda C|$ is an $(n-k-1)$-cell. Let F be a cone on $|\lambda C|$; we then use the standard trick which makes C^*, which was the cone on $|Lk(\eta C, d^2\mathscr{P})|$, which is equivalent to $\partial C * |\lambda C|$, equivalent to $C \times F$:

$$g : C^* \approx C \times F,$$

in which the set $C^* \cap \{\partial C\}^*$, which was mapped to $N_{d\{pC\}*\lambda C}(d\{\partial C\})$, **198**
corresponds to

$$g(C^* \cap \{\partial C\}^*) = (\partial C) \times F.$$

This shows, from our second way of looking at handles, that $(C^*, C^* \cap \{\partial C\}^*)$ is a k-handle.

We might remark that in case (i), $C \cap \partial(C^*)$ is an attaching sphere, but that in case (ii), this lies in the boundary of the attaching tube; that is why case (ii) is somewhat more complicated than case (i). □

8.3 Statement of the theorems, applications, comments

Here we state the main theorems of handle-theory and apply them to situations such as s-cobordism and unknotting. We outline the proofs, so that the rest of our work is devoted to the techniques which make this outline valid. We say a few words about gaps (such as a thorough discussion of Whitehead torsion) for which there are adequate references. Our theorems and proofs are quite similar to those well-known for differential manifolds; of course, there is no worry about rounding off corners; there is no need to use isotopy-extension theorems, since cellular moves suffice. Finally, the crucial point is for homotopy to imply isotopy in certain unstable dimensions; the result needed here has been described by Weber, [see C. Weber, L'élimination des points doubles dans le cas combinatoire, Comm. Math. Helv., Vol.41, Fasc 3, 1966-67]; for variety and interest, we prove the necessary result in a quite different way

199 **Definition 8.3.1.** A relative n-manifold (M, X) is said to be *geometrically trivial,* if $M \searrow X$.

If (M, X) is a special case, where X is an $(n-1)$-submanifold of ∂M, M and n-manifold, when geometric triviality means just that $M \approx X \times I$ with X corresponding to $X \times 0$.

When $A \subset B$ are finite CW-complexes, with $A \hookrightarrow B$ a homotopy equivalence, the *torsion* of (B, A), denoted by $\tau(B, A)$, is a certain element of the Whitehead group of $\pi_1(B)$.

Definition 8.3.2. Suppose (M, X) is a special case. That (M, X) is *algebraically trivial* means:

(1) $X \hookrightarrow M$ is a homotopy equivalence.

(2) $\tau(M, X) = 0$

(3) $\partial(M, X) \hookrightarrow M$ induces an isomorphism on π_1.

[Remark: Using a form of Lefschetz duality in the universal covering spaces, it is provable that (3) is implied by (1) plus the weaker condition that $\partial(M, X) \hookrightarrow M$ induces an injection on π_1].

If (M, X) is not a special case, let N be a regular neighbourhood of X in M. Define $M_1 = M - N$, and $X_1 = bd_M N$. Then (M_1, X_1) is a special case, uniquely determined, upto polyhedral equivalence, by (M, X). We call (M, X) algebraically trivial whenever (M_1, X_1) is algebraically trivial.

When we know of (M, X) that only conditions (1) and (3) are satisfied, (M, X) being special, we call (M, X) an *h-cobordism,* and $\tau(M, X)$ the *torsion* of this h-cobordism.

200 Clearly, if (M, X) is geometrically trivial, it is also algebraically trivial. The converse, we shall show, is true for relative n-manifold, $n \geq 6$.

Let (M, X) be a relative n-manifold which is a special case. Here are the main results.

Theorem A. *If (M, X) is 1-connected, and $\ell \leq n - 4$, then (M, X) has a handle presentation with no handles of indices $\leq \ell$. If furthermore, (M, X) has a handle presentation with handles of indices $\leq p$ only,* *then it has a handle presentation with handles of indices* $\geq \ell + 1$ *and* $\leq \mathrm{Max}(\ell + 2, p)$ *only.*

Theorem B. *If* (M, X) *has a handle presentation with handles of indices* $\leq n - 3$ *only, and* $n \geq 6$*, and if* $X \hookrightarrow M$ *is a homotopy equivalence with* $\tau(M, X) = 0$*, then it has a presentation without any handles; so that* $M \searrow X$.

Theorem C. *If* (M, X) *is algebraically trivial and* $n \geq 6$*, then it is geometrically trivial.*

Theorem C holds for the general relative n-manifold, and this follows from Theorem C in the special case by referring to the special case (M_1, X_1) described earlier.

Theorem A and B imply Theorem C by *duality*, which is described in 8.8. We start with a handle presentation $\mathscr{H}$ of (M, X); by Theorem A we can change the dual presentation $\mathscr{H}^*$ into one with no handles of index $\leq n - 4$; dualizing this, we get a handle presentation $\mathscr{H}_1$ of (M, X) without handles of indices ≥ 4; since $n \geq 6$, Theorem B applies to $\mathscr{H}_1$.

8.3.3 We now list the techniques used in proving Theorems A and B.

(1) *Cancelling pairs of handles.* In a handle presentation $\mathscr{H} =$ **201**
$(A_{-1}, \ldots, A_n)$, sometimes there is a very explicit geometrical reason why a $(k - 1)$-handle $\mathfrak{h}$ and a k-handle $\mathfrak{K}$ nullify each other, so that they can be dropped from the handle presentation. If N is the transverse tube of $\mathfrak{h}$, and T the attaching tube of $\mathfrak{K}$, and $N - N \cap T$ and $T - N \cap T$ are both $(n - 1)$-cells, this is the case. This alone suffices to prove Theorem A when $\ell = 0$. We discuss this in 8.5.

(2) *Modifying the handle presentation.* We want to shrink down transverse and attaching tubes until they become manageable, and to isotop things around. This can be done without damaging the essential structure, which consists of (a) The polyhedral equivalence class of (M, X), (b) The number of handles of each index, (c) The salient features of the algebraic structure, namely, the maps $\pi_k(A_k, A_{k-1}) \to \pi_{k-1}(A_{k-1}, A_{k-2})$ and bases of these groups. This is done in 8.4.

(3) *Inserting cancelling pairs of handles,* the opposite to (1) is sometimes necessary in order to simplify the algebraic structure; this occurs in 8.6. This, together with (1) and (2), allows us to prove Theorem A for $\ell = 1$, at the expense of extra 3-handles. Once we have done this, there are no more knotty group-theoretic difficulties, and the universal covering spaces of the A_i's are all embedded in each other. Then we can take a closer look at.

(4) *The algebraic structure.* This consists of the boundary maps $\pi_k(A_k, A_{k-1}) \to \pi_{k-1}(A_{k-1}, A_{k-1})$. When there are no 1-handles, these groups are free modules over the fundamental-group-ring, with bases
202 determined, upto multiplying by $\pm\pi$, by the handles. We can change bases in certain prescribed ways by inserting and cancelling pairs of handles. This allows us to set up a situation where a $(k-1)$-handle and a k-handle *algebraically* cancel. We discuss this in 8.9. And now, both Theorems A and B follow if we can get algebraically cancelling handles to cancel in the real geometric sense. This amounts to getting an *isotopy* out of a *homotopy* of attaching spheres; this is, of course, the whole point; all the other techniques are a simple translation to handle presentations of the theory of simple homotopy types of J.H.C. Whitehead.

(5) *The isotopy lemma.* This is the point where all dimensional restrictions really make themselves felt. The delicate case, which applies to $(n-3)$-and $(n-4)$-handles, just barely squeaks by.

8.3.4 The *s*-cobordism theorem. By an *s-cobordism* is meant a triple $(M; A, B)$, where M is an n-manifold; A and B are disjoint $(n-1)$-submanifolds of ∂M; $\partial M - A \cup B$ is polyhedrally equivalent to $\partial A \times I$ in such a way that ∂A corresponds to $\partial A \times 0$ (and, of course, ∂B to $\partial A \times 1$); $A \hookrightarrow M$ and $B \hookrightarrow M$ are homotopy equivalences; and $\tau(M, A) = 0$.

A *trivial cobordism* is a triple $(M; A, B)$ equivalent to $(A \times I; A \times 0, A \times 1)$.

Theorem . *If* $(M; A, B)$ *is an s-cobordism, and* $\dim M \geq 6$, *then* $(M; A, B)$ *is a trivial cobordism.*

Proof. This follows from theorem C. The pair (M, A) is a relative manifold, special case; and all the hypothesis of Theorem C are clearly valid;

in particular, $\pi_1(B) \approx \pi_1(\partial M - A) \approx \pi_1(M)$, since $B \hookrightarrow M$ is a homotopy equivalence. Hence, by theorem C, (M, A) is equivalent to $(A \times I, A \times 0)$. We know, by assumption, that $\overline{\partial M - A \cup B}$ is a regular neighbourhood of ∂A in $\overline{\partial M - A}$; and clearly $\partial A \times I$ is a regular neighbourhood of $\partial A \times 0$ in $\overline{\partial(A \times I) - A \times 0}$. Thus we can fix up the equivalence of (M, A) to $(A \times I, A \times 0)$ to take $\overline{\partial M - A \cup B}$ onto $\partial A \times I$; this leaves B to map onto $A \times 1$, which shows the cobordism is trivial. □ 203

We remark that if $\pi_1(A)$ is trivial, then $\tau(M, A) = 0$ automatically. It is with this hypothesis that Smale originally proved his theorem; various people (Mazur and Barden) noticed that the hypothesis needed in the non-simply-connected case, was just that $A \hookrightarrow M$ he a simple homotopy equivalence (whence the "s"); i.e. $\tau(M, A) = 0$.

8.3.5 Zeeman's unknotting theorem. We have already described the notion of an unknotted sphere.

Theorem . *If $A \subset B$, where A is a k-sphere, B an n-sphere, and $k \leq n-3$, then A is unknotted in B.*

Proof. By induction on n. For $n \leq 5$, the cases are all quite trivial, except for $k = 2$, $n = 5$, which has been treated earlier. For $n \geq 6$ we will show that the pair (B, A) is equivalent to the suspension of (B', A') where B' is an $(n-1)$-sphere and A' a $(k-1)$-sphere; and clearly the suspension of an unknotted pair of spheres is unknotted.

To desuspend, for $n \geq 6$, we proceed thus:

If $x \in A$, then the link of x in (B, A) is a pair of spheres which is unknotted, by the inductive hypothesis. That is to say, A is *locally unknotted* in B. In particular, we can find an n-cell $E \subset B$, such that 204
$E \cap A$ is a k-cell unknotted in E; and so that $(\partial E, \partial(E \cap A))$ is bicollared in (B, A); in fact, this could have been done whether or not A were locally unknotted.

Define $F = \overline{B - E}$. By earlier results F is an n-cell. Consider the relative manifold $(F, F \cap A)$. It is easily seen that this pair is algebraically trivial; because of codimension ≥ 3 all fundamental groups are trivial, and so Whitehead torsion is no problem; the homology situation is an easy exercise in Alexander duality. Hence, by Theorem C, F collapses to $F \cap A$; since $F \cap A$ is a k-cell, it collapses to a point; putting these together, $(F, F \cap A)$ collapses to (pt, pt) in the category of pairs; these collapses are homogeneous in the pair (B, A) because of local unknottedness. We started with $(F, F \cap A)$ bicollared, and hence, by the regular neighbourhood theorem, suitably stated for pairs, $(F, F \cap A)$ is a regular neighbourhood of $x \in A$ in (B, A), which is an unknotted cell pair (again using local unknottedness).

Thus (B, A) is the union of two unknotted cell pairs $(E, E \cap A)$ and $(F, F \cap A)$, which shows it is polyhedrally equivalent to the suspension of $(\partial E, \partial(E \cap A))$. □

Remark 1. This is just Zeeman's proof, except that we use our Theorem C where Zeeman uses the cumbersome technique of 'sunny collapsing".

Remark 2. Lickorish has a theorem for desuspending general suspensions embedded in S^n in codimension ≥ 3. It is possible, by a similar
205 argument, to replace "sunny collapsing" by Theorem C. The case $n = 5$ can be treated by a very simple case of sunny collapsing.

Remark 3. If $A \subset B$, where A is an $(n-2)$-sphere locally unknotted in the n-sphere B, and $n \geq 6$, and $B - A$ has the homotopy type of a 1-sphere, then A is unknotted in B.

Proof exactly as in the codimension 3 case; we need to know that Whitehead torsion is OK, which it is since the fundamental group of the 1-sphere is infinite cyclic; and this group has zero Whitehead torsion, by a result of Graham Higman (units of group-rings).

Remark 4. It has been a folk result for quite a while that the Unknotting Theorem followed from the proper statement of the h-cobordism theorem.

8.3.6 Whitehead torsion. For any group π, there is defined a commutative group $Wh(\pi)$. Elements of $Wh(\pi)$ are represented by square, invertible matrices over the integer group ring $Z\pi$. Two matrices A and

B represent the same element in $Wh(\pi)$, if and only if there are identity matrices I_k and I_ℓ, and a product E of elementary matrices, so that $A \oplus I_K = E.\ (B \oplus I_\ell)$. Here

$$I_k \quad \text{denotes the } k \times k \text{ identity matrix.}$$

$$U \oplus V = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

An elementary matrix is one of the following:

(a) $I_n + e_{ij}$, where e_{ij} is the $n \times n$ matrix all of whose entries are zero except for the ij^{th} which is 1; and $i \neq j$.

(b) $E_n(\alpha; K)$, which is the $n \times n$ matrix equal to the identity matrix, except that the kk^{th} entry is α; we restrict α to be an element of $\pm\pi$.

By cleverly composing matrices of this sort, we can obtain $I_n + \lambda e_{ij}$ **206**
for any $\lambda \in Z\pi$, for instance.

Addition in $Wh(\pi)$ is induced from $\oplus$, or, equivalently, from matrix multiplication.

The geometric significance of $Wh(\pi)$ is that a homotopy equivalence $f : K \to L$ between finite CW-complexes determines an element of $Wh(\pi)$, $\pi = \pi_1(K)$, called the *torsion* of f. If the torsion is zero, wonderful things (e.g. s-cobordism) happen.

If π is the trivial group, then $Wh(\pi) = 0$, basically because $Z\pi$ is then a Euclidean domain.

If π is infinite cyclic, then $Wh(\pi) = 0$, by Higman. His algebraic argument is easily understood; it is, in some mystical sense, the analogue of breaking something the homotopy type of a circle into two contractible pieces on which we use the result for the trivial group.

If π has order 5, then $Wh(\pi) \neq 0$. In fact, recent computations show $Wh(\pi)$ to be infinite cyclic.

Various facts about Wh can be found in Milnor's paper. ["Whitehead Torsion" Bulletin of A.M.S., Vol. 72, No. 3, 1966]. In particular, the torsion of an h-cobordism can be computed (in a straight-forward, may be obvious, way) from any handle presentation.

There is another remark about matrices that is useful. Let A be an $n \times k$ matrix over $Z\pi$, such that any k-row-vector [i.e. $1 \times k$ matrix] is some left linear combination, with coefficients in $Z\pi$, of the rows of A;
207 in other words, A corresponds to a *surjection* of a free $Z\pi$-module with n basis elements onto one with k basis elements. Let 0_k denote the $k \times k$ zero matrix. Then there is a product of elementary $(n+k) \times (n+k)$ matrices, E, so that $E.\begin{pmatrix} A \\ 0_k \end{pmatrix} = \begin{pmatrix} I_k \\ 0_{n \times k} \end{pmatrix}$. This is an easy exercise.

8.3.7 In homotopy theory we shall use such devices as univesal covering spaces, the relative Hurewicz theorem, and some homology computations (with infinite cyclic coefficient group). For example, if (H, T) is a k-handle, then

$$\begin{aligned} &H_i(H, T) = 0 \quad \text{for} \quad i \neq h \\ &H_k(H, T) = Z, \quad \text{an infinite cyclic group,} \end{aligned}$$

We always arrange to have the fundamental group to act on the left on the homology of the universal covering space.

Suppose that (M, X) is a relative n-manifold, special case, and $(N, X) = (M, X) + \mathfrak{h}_1 + \cdots + \mathfrak{h}_p$, where the $\mathfrak{h}$'s are handles of index k. Suppose X, M, N are connected, and that $\pi_1(M) \to \pi_1(N)$ is an isomorphism; this implies that we can imagine not only that $M \subset N$, but that $\tilde{M} \subset \tilde{N}$, where "$\sim$" denotes universal covering space. Call $\pi = \pi_1(M)$.

Then the homology groups $H_i(\tilde{N}, \tilde{M})$ are left $Z\pi$-modules. More explicitly, $H_i(\tilde{N}, \tilde{M}) = 0$ if $i \neq k$; and $H_k(\tilde{N}, \tilde{M})$ is a free $Z\pi$-module with basis $\{[\mathfrak{h}_1], \ldots, [\mathfrak{h}_p]\}$. What does $[\mathfrak{h}_j]$ mean? We take any lifting of $\mathfrak{h}_j = (H, T)$ to a handle (H', T') in $\tilde{N}$; we pick either generator of $H_k(H', T')$, and map into $H_k(\tilde{N}, \tilde{M})$ by inclusion, the result is $[\mathfrak{h}_j]$. The
208 ambiguity in defining $[\mathfrak{h}_j]$ is simply stated: If we make another choice, then instead of $[\mathfrak{h}_j]$ we have $\alpha[\mathfrak{h}_j]$, where $\alpha \in \pm\pi$.

When $k \geq 2$, we can go further and say that, by the relative Hurewicz theorem, $\pi_k(\tilde{N}, \tilde{M}) \approx H_k(\tilde{N}, \tilde{M}) \approx \pi_k(N, M)$. And thus we have a fairly well-defined basis of $\pi_k(N, M)$ as a 2π-module, dependent on the handles $\mathfrak{h}_1, \ldots, \mathfrak{h}_p$.

This shows, by the way, that (N, M) is $(k-1)$-connected. We might have expected this, since, homotopically, N is obtained from M by attaching k-cells.

Another thing is a version of Lefschetz duality as follows: If M is an oriented manifold, and $X \subset M$ with X a polyhedron, then $H^i(M, X) \approx H_{n-i}(M, \partial M - X)$. Since universal covering spaces can all be oriented, this works there. In particular, if $X \hookrightarrow M$ is a homotopy equivalence, then $H^i(\tilde{M}, \tilde{X}) = 0$ for all i, and so $H_i(\tilde{M}, \partial\tilde{M} - \tilde{X}) = 0$ for all i. When $\partial\tilde{M} - \tilde{X}$ is the universal covering space of $\partial M - X$, that is, when $\pi_1(\partial M - X) \approx \pi_1(M)$, then the relative Hurewicz theorem will show that $\partial M - X \hookrightarrow M$ is a homotopy equivalence.

8.3.8 Infinite polyhedra. An *infinite polyhedron P* is a locally compact subset of some finite-dimensional real vector space, such that for every $x \in P$, there is an ordinary polyhedron $Q \subset P$, such that x is contained in the topological interior of Q in P. A *polyhedral map* $f : P_1 \to P_2$, between infinite polyhedra is a function, such that for every ordinary polyhedron $Q \subset P_1$, the graph $\Gamma(f|Q)$ is an ordinary polyhedron.

The category of infinite polyhedra includes ordinary polyhedra; and **209**
in addition, every open subset of an infinite polyhedron is an infinite polyhedron.

The link of a point in an infinite polyhedron is easily defined; it turns out to be a polyhedral equivalence class of ordinary polyhedra. Hence the notions of manifold any boundary in this setting are easily defined.

If M is an infinite n-manifold, then any compact subset $X \subset M$ is contained in the topological interior of some ordinary n-manifold $N \subset M$.

As for isotopies, we restrict ourselves to isotopies which are the identity outside some compact set; such are the isotopies obtained from finitely many cellular moves. Any such isotopy on the boundary of M can be extended to an isotopy of this sort on M.

We can talk of regular neighbourhoods of ordinary (= compact) subpolyhedra in an infinite polyhedron, and the same theorems (including isotopy, in this sense) hold.

These concepts are useful here because if (M, X) is a relative n-manifold, then $\partial(M, X)$ is an infinite n-manifold. And now, any isotopy of $\partial(M, X)$ extends to an isotopy of M, leaving a neighbourhood of X fixed. In other words, this is convenient language for dealing with relative manifolds. This is the only situation where we shall speak of infinite polyhedra; it is, of course, obvious that infinite polyhedra can be of use in many other cases which are not discussed in these notes (in particular, in topological applications of the "Engulfing Theorem").

8.4 Modification of handle presentations

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If $\mathscr{X} = (A_{-1}, \ldots, A_n)$ and $\mathscr{H}' = (B_{-1}, \ldots, B_n)$ are handle presentations of the relative n-manifolds (M, X) and (M^1, X^1) respectively, an *isomorphism* between $\mathscr{H}$ and $\mathscr{H}^1$ is a polyhedral equivalence $h : M \to N$ taking X onto X' and A_i onto B_i for all i. Such an isomorphism gives a $1-1$ function between handles and preserves various other structures.

Let $\mathscr{H} = (A_{-1}, \ldots, A_n)$ be a handle presentation of the relative n-manifold (M, X) and let $f : A_k \to A_k$ be a polyhedral equivalence taking X onto itself. Then by $\mathscr{H}_f$ is meant the handle presentation $(B_{-1}, \ldots, B_n)$ of (M, X), where

$$\begin{aligned} B_i &= f(A_i) \quad \text{for} \quad i < k \\ B_k &= f(A_k) = A_k \\ B_i &= A_i \quad \text{for} \quad i > k. \end{aligned}$$

It is clear that $\mathscr{H}_f$ is a handle presentation of (M, X), the handles of index $> k$ are equal to those of $\mathscr{H}$, while a handle of $\mathscr{H}$ of index $\leq k$ will correspond via f to a handle of $\mathscr{H}_f$.

There is another way upto isomorphism of looking at $\mathscr{H}_{f^{-1}}$.

Suppose $f : A_k \to A_k$ is as before. Let (H, T) be a $(k+1)$-handle of the presentation $\mathscr{H}$. Attach (K, T) to A_k not by the inclusion of T in $\partial(A_k, X)$, but by $f|T$. In this way attaching all $(k+1)$-handels we get a relative manifold (B_{k+1}, X) and an equivalence f_{k+1} extending f. Similarly attach the $(k+2)$-handles to B_{k+1} one for each $(k+2)$-handle of A_{k+2} by the map f_{k+1} suitably restricted; and so on. In this way we get a relative
211 manifold (B_n, X) and a handle presentation $(A_{-1}, \ldots, A_k, B_{k+1}, \ldots, B_n)$ of (B_n, X). This will be denoted by $\mathscr{H}^f$. f_n gives an equivalence of (M, X) with (B_n, X) and an isomorphism of $\mathscr{H}_{f^{-1}}$ with $\mathscr{H}^f$.

The main use in this chapter of the above modifications is for simplifying handle presentations, that is to obtain presentations with as few handles as possible, or without any handles or without handles upto certain index using the given algebraic data about (M, X). It should be noted that (1) $\mathscr{H}_f$ need not be isomorphic to $\mathscr{H}$ and (2) $\mathscr{H}^f$ is not a handle presentation of (M, X). (2) is not a serious drawback, since $\mathscr{H}^f$ isomorphic to $\mathscr{H}_{f^{-1}}$ via f_n^{-1} and so whatever simplification one can do for $\mathscr{H}^f$ can be done also for $\mathscr{H}_{f^{-1}}$, which is a handle presentation of (M, X) or we can first do the simplifications in $\mathscr{H}^f$ and pull the new handle presentation to one of (M, X) by f_n^{-1}. We will adopt the procedure which is convenient in the particular case. If $f : A_k \to A_k$ is isotopic to the identity leaving X fixed, (and this will be usually the case), then $\mathscr{H}$ and $\mathscr{H}_f$ will have many homotopy properties in common; but more of this later.

The most frequently used ways of modifications are catalogued below:

8.4.1 Let (H, T) be a k-handle of the presentation $\mathscr{H} = (A_{-1}, \dots, A_n)$ of (M, X). Then if S is a transverse sphere and $N = \overline{\partial H - T}$ the transverse tube, we have N a regular neighbourhood of S in $\partial(A_k, X)$. If N' is any other regular neighbourhood of S in $\partial(A_k, X)$, there is an isotopy of $\partial(A_k, X)$ relating N and N' and this can be extended to A_k, to give an **212**
end result f_1, with $f_1(N) = N'$. Then $\mathscr{H}_{f_1}$ has its new handle $(f_1 H, f_1 T)$ whose transverse tube in N'.

8.4.2 Let (H_1, T_1) be a $(k+1)$-handle of $\mathscr{H}$, with an attaching sphere $\sum$. Then T_1 is a regular neighbourhood of $\sum$ in $\partial(A_k, X)$. If T_1' is any other regular neighbourhood of $\sum$ in $\partial(A_k, X)$ we can obtain a polyhedral equivalence f_2 of A_k which isotopic to 1 fixing X, such that $f_2(T_1) = T_1'$. Then $\mathscr{H}^{f_2}$ (which is isomorphic to $\mathscr{H}_{f_2^{-1}}$) will have its $(k+1)$-handle corresponding to (H_1, T_1) to have attaching tube T_1', and handles of index $\leq k$ will be unchanged.

Combining 8.4.1 and 8.4.2, we have

Proposition 8.4.3. *Let $\mathscr{H} = (A_{-1}, \dots, A_n)$ be a handle presentation of the relative n-manifold (M, X); let $\mathfrak{h}$ be a k-handle and $\mathfrak{K}$ a $(k+1)$-*

handle with a transverse sphere of $\mathfrak{h}$ being S and an attaching sphere of $\mathfrak{K}$ being $\sum$. Let N and T be regular neighbourhoods of S and $\sum$ in $\partial(A_k, X)$. Then there is a handle presentation $\mathscr{H}'$ of (M', X') is equivalent to (M, X) with $\mathscr{H}'$ being isomorphic to $\mathscr{H}_f$ for some $f : A_k \to A_k$ isotopic to the identity leaving X fixed; so that in $\mathscr{H}'$ the handles $\mathfrak{h}'$ and $\mathfrak{K}'$ corresponding to $\mathfrak{h}$ and $\mathfrak{K}$ are such that:

the transverse tube of $\mathfrak{h}'$ is N,

the attaching tube of $\mathfrak{K}'$ is T, and

the k^{th} level A'_k of $\mathscr{H}'$ is equal to the

k^{th} level A_k of $\mathscr{H}$.

Proof. Using the equivalences f_1 and f_2 given by 8.4.1 and 8.4.2,
213 $(\mathscr{H}_{f_1})^{f_2}$ is the required presentation. It is isomorphic to the presentation $\mathscr{H}_{(f_2^{-1} f_1)}$ of (M, X). Since both f_1 and f_2 are isotopic to the identity leaving X fixed, $f_2^{-1} f_1$ has the same property. The last point is obvious. □

8.4.4 Let $\mathfrak{K} = (H, T)$ be a $(k+1)$-handle of $\mathscr{H}$, and S an attaching sphere of $\mathfrak{K}$. S is in $\partial(A_k, X)$. Suppose that S' is another k-sphere in $\partial(A_k, X)$ and that there is an equivalence f of A_k taking X onto itself and such that $f(S) = S^1$. Then in $\mathscr{H}^f$, the handle $\mathfrak{K}^1$ corresponding to $\mathfrak{K}$ will have S^1 as an attaching sphere. If, for example, we can go from S to S^1 by cellular moves, the we can obtain an equivalence f of A_k isotopic to 1 leaving X fixed and with $f(S) = S^1$. This will also be used in cancellation of handles, where it is more convenient to have certain spheres as attaching spheres than the given ones.

8.4.5 Let $\mathfrak{h}$ be a k-handle in a handle presentation $\mathscr{H} = (A_{-1}, \ldots, A_n)$ of a relative n-manifold. Then if $k \leq n-2$, there is h isotopic to the identity, $h : A_k \to A_k$, leaving X fixed, such that the handle $\mathfrak{h}^1$ of $\mathscr{H}_h$ corresponding to $\mathfrak{h}$ has a boundary core in $\partial(A'_{k+1}, X)$ where $\mathscr{H}_h = (A'_{-1}, \ldots, A'_n)$.

(Reader, have faith that this is usefull)

We prove this by choosing attaching spheres for all the $(k+1)$-handles, finding a transverse sphere for $\mathfrak{h}$ that intersects all the attaching spheres only finitely, noting that the transverse sphere contains other points, and then shrinking the attaching tubes and the transverse tube conveniently. More explicitly. **214**

Let $\mathfrak{K}_1, \ldots, \mathfrak{K}_p$ be the $(k-1)$-handles of $\mathscr{H}$, with attaching spheres $S_1, \ldots, S_p$. Let N be the transverse tube of $\mathfrak{h}$; there is a polyhedral equivalence $f : N \approx D^k \times \Delta^{n-k}$, so that for any $x \in \operatorname{int} D^k$, $f^{-1}(k \times \partial\Delta^{n-k})$ is a transverse sphere to $\mathfrak{h}h$. Now, $(S_1 \cup \ldots \cup S_p) \cap N$ is $(\leq k)$-dimensional, and so, triangulating $\mathrm{proj}_D k \cdot f$ so as to be simplicial and picking x in the interior of a k-simplex of D^k (see 4.2.14), we have found a transverse sphere $\sum = f^{-1}(x \times \partial\Delta^{n-k})$ to $\mathfrak{h}$, such that $\sum \cap (S_1 \cup \ldots \cup S_p)$ is finite. Now, $\sum$ is an $(n-k-1)$-sphere; and since $k \leq n-2$, contains infinitely many points; there is $y \in \sum - (S_1 \cup \ldots, \cup S_p)$.

Now then, if we take very thin regular neighbourhoods of $\sum$, S_1, $\ldots, S_p$ in $\partial(A_k, X)$ the regular neighbourhood of $\sum$ will intersect those of $S_1, \ldots, S_p$ in only small cells near each point of intersection of $\sum \cap (S_1$, $\ldots, S_p)$, and hence there will be a cross-section of the $\sum$-neighbourhood [i.e., corresponding to $D^k \times z$, $z \in \partial\Delta^{n-k}, (x, z) = f(y)$], through y, not meeting any of the S_i neighbourhoods. We make these regular neighbourhoods the transverse tube of $\mathfrak{h}$ and the attaching tubes of $\mathfrak{K}_1, \ldots, \mathfrak{K}_p$, by changing $\mathscr{H}$ to $(\mathscr{H}_{g_1})^{g_2}$, where g_1 and g_2 are equivalences $A_k \to A_k$ isotopic to the identity, fixing X. In $(\mathscr{H}_{g_1})^{g_2}$ we have a boundary core of the handle corresponding to $\mathfrak{h}$ which misses all the attaching tubes of the $(k+1)$-handles (this is that "cross-section through y"). We define $h = g_2^{-1} g_1$: and since $\mathscr{H}_h$ is isomorphic to $(\mathscr{H}_{g_1})^{g_2}$, we have some boundary core of $\mathfrak{h}^1$ when $\mathfrak{h}^1$ is the handle corresponding to $\mathfrak{h}$ which **215**
does not intersect the attaching tubes of all the $(k+1)$-handles, and is therefore in $\partial(A'_{k+1}, X)$.

8.5 Cancellation of handles

Convention: Let us make the convention that a submanifold of another manifold should mean this:

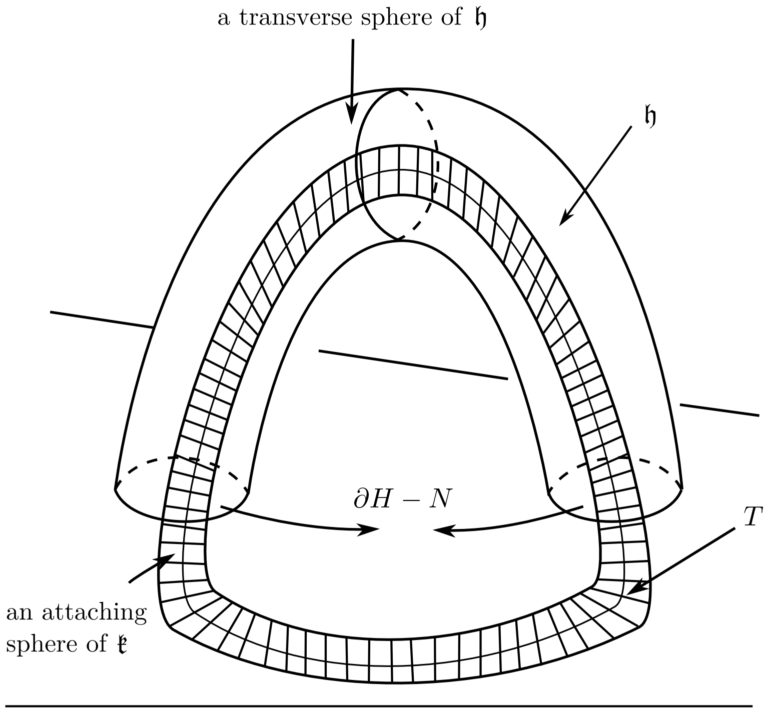
If $A \subset B$, A and B are PL-manifolds, we call A a submanifold of B, if and only if, $A \cap \partial B$ is $(\dim A - 1)$-submanifold of ∂A. We are usually in this section interested only in the case $\dim A = \dim B$.

With this convention, if A is a submanifold of B, then $\overline{B - A}$ is a submanifold of B, and $bd_B(A) = bd_B(\overline{B - A}) = \overline{\partial A - \partial B}$. If $C \subset B \subset A$ all PL-manifolds such that each is a submanifold of the next, then $A - (\overline{B - C}) = \overline{A - B} \cup C$. We may therefore be justified somewhat in writing $A - B$ for $\overline{A - B}$.

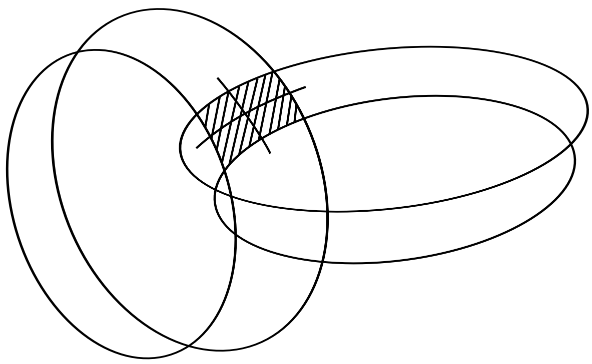
Thus, hereafter, A is a submanifold of B means that A is a submanifold of B in the above sense, and in that case $B - A$ stands for $\overline{B - A}$.

Let $\mathscr{H} = (A_{-1}, \ldots, A_n)$ be a handle presentation of a relative n-manifold (M, X). Let $\mathfrak{h} = (H, \partial H - N)$ be a k-handle with transverse tube N, and $\mathfrak{K} = (K, T)$ be a $(k+1)$-handle with attaching tube T. Note that $N \cup T \subset \partial(A_k, X)$ with the above conventions we can write

$$A_k + \mathfrak{h} = ((A_k - \mathfrak{h}) \cup K) \cup H.$$



$\mathfrak{k}$ is attached to A_k by T



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Definition 8.5.1. We say that $\mathfrak{h}$ and $\mathfrak{K}$ can be *cancelled* if 217

(1) $N \cap T$ is a submanifold of both N and T

(2) $N - (N \cap T)$ and $T - (N \cap T)$ are both $(n-1)$-cells.

Suppose $\mathfrak{h}$ and $\mathfrak{K}$ can be cancelled. Then

Assertion 1. $(A_k - \mathfrak{h}) \cap K$ is an $(n-1)$-cell contained in $\partial(A_k - \mathfrak{h}, X)$ and in ∂K.

In fact, $(A_k - \mathfrak{h}) \cap K = T - (N \cap T)$, which we assumed to be an $(n-1)$-cell.

Assertion 2. $((A_k - \mathfrak{h}) \cup K) \cap H$ is an $(n-1)$-cell contained in $\partial((A_k - \mathfrak{h}) \cup K, X)$ and in ∂H. For

$$\begin{aligned}((A_k - \mathfrak{h}) \cup K) \cap H &= \text{attaching tube of } \mathfrak{h} \text{ plus } N \cap T\\ &= (\partial H - N) \cup (N \cap T)\\ &= \partial H - (N - N \cap T)\end{aligned}$$

and this is an $(n-1)$-cell, since ∂H is an $(n-1)$-sphere and $(N - N \cap T)$ is an $(n-1)$-cell in ∂H.

Combining these two assertions with proposition 8.2.1, we have

Proposition 8.5.2. *Suppose $\mathscr{H} = (A_{-1}, \ldots, A_n)$ is a handle presentation of a relative n-manifold (M, X); and there are $\mathfrak{h} = (H, \partial H - N)$ a k-handle, and $\mathfrak{K} = (K, T)$ a $(k+1)$-handle that can be cancelled. Let $\mathscr{U}$ be any neighbourhood of $N \cap T$ in A_k. Then there is a polyhedral equivalence*

$$f : (A_k - \mathfrak{h}, X) \approx (A_k + \mathfrak{K}, X)$$

which is identity outside $\mathscr{U}$.

218 This being so, we construct a new-handle presentation $(B_{-1}, \ldots, B_n)$ of (M, X), which we denote by $\mathscr{H} - (\mathfrak{h}, \mathfrak{K})$ as follows:

$$\begin{aligned}B_i &= f(A_i) \quad \text{for} \quad i < k\\ B_k &= f(A_k - \mathfrak{h}) = A_k + \mathfrak{K}\\ B_i &= A_i \quad \text{for} \quad i > k.\end{aligned}$$

This of course depends on f somewhat observe that, since the attaching tubes of the $(k+1)$-handles are disjoint, the attaching tubes of $(k+1)$-handles other than $\mathfrak{K}$ are in $\partial(A_k, X) - T \subset \partial(A_k + \mathfrak{K}, x)$, so that $\mathscr{H} - (\mathfrak{h}, \mathfrak{K})$ is a genuine handle-presentation.

8.5.3 (Description of $\mathscr{H} - (\mathfrak{h}, \mathfrak{K})$). The number of i-handles in $\mathscr{H} - (\mathfrak{h}, \mathfrak{K})$ is the same as the number of i-handles of $\mathscr{H}$ for $i \neq k, k+1$. For $i > k$, each i-handle of $\mathscr{H}$ is a i-handle of $\mathscr{H} - (\mathfrak{h}, \mathfrak{K})$ with the single exception of $\mathfrak{K}$; and conversely. For $i \leq k$, each i-handle of $\mathscr{H}$ except $\mathfrak{h}$, say ℓ, corresponds to the i-handle $f(\ell)$ of $\mathscr{H} - (\mathfrak{h}, \mathfrak{K})$ and conversely each i-handle of $\mathscr{H} - (\mathfrak{h}, \mathfrak{K})$ is of this form. If the attaching tube of $\mathfrak{K}$ does not interset some k-handle ℓ, we can arrange $f|\ell$ to be identity, so that ℓ itself occurs in $\mathscr{H} - (\mathfrak{h}, \mathfrak{K})$.

The conditions for $\mathfrak{h}$ and $\mathfrak{K}$ to cancel are somewhat stringent. We now proceed to obtain a sufficient condition on $\mathfrak{h}$ and $\mathfrak{K}$, which will enable us to cancel the handles corresponding to $\mathfrak{h}$ and $\mathfrak{K}$ in some $\mathscr{H}_f$. This requires some preliminaries.

Suppose A, B, C are three PL-manifolds, $A \cup B \subset C - \partial C$. $\dim A = p$, $\dim B = q$ and $\dim C = p + q$, $\partial A = \partial B = \emptyset$. Let $x \in A \cap B$.

Definition 8.5.4. A and B are said to intersect *transversally at x in C*, **219**
if there is a neighbourhood F of x in C and a polyhedral equivalence $f : F \xrightarrow{\approx} S * \sum * v$ where S is a $(p-1)$-sphere, $\sum$ a$(q-1)$-sphere, such that

(1) $f(x) = v$

(2) $f(A \cap F) = S * v$

(3) $f(B \cap F) = \sum * v$.

Proposition 8.5.5. *Let S and $\sum$ be $(p-1)$-and $(q-1)$-spheres respectively and $E = S * \sum * v$. Let $D = S * v$, $\Delta = \sum * v$. Suppose $\mathscr{E}$ is any simplicial presentation of E containing full subpresentations $\mathscr{D}$ and $\mathscr{A}$ covering D and Δ. Let $P = |N_{\mathscr{E}}(\mathscr{D})|$ and $Q = |N_{\mathscr{E}}(\mathscr{A})|$. Then*

(1) $P \cap Q$ is a submanifold both of P and Q and is contained in the interior of E (P, Q and $P \cap Q$ are all $(p+q)$-manifolds)

(2) $P - P \cap Q \searrow P \cap \partial E$

$Q - P \cap Q \searrow Q \cap \partial E.$

Proof. First observe that, if the proposition is true for some centering of $\mathscr{E}$, then it is true for any centering of $\mathscr{E}$. Next, if $\mathscr{E}'$ is some other presentation of E such that D and Δ are covered by full subpresentations, it is possible to choose centerings of $\mathscr{E}$ and $\mathscr{E}'$ so that $P = P'$ and $Q = Q'$. (P', Q' denoting the analogoues of P and Q with reference to $\mathscr{E}'$). Thus it is enough to prove the proposition for some suitable presentation $\mathscr{E}'$ of E and a suitable centering of $\mathscr{E}'$. Now we choose $\mathscr{E}'$
220 to be a join presentation of $E = S * \sum * v$ and choose the centering so that (see 6.8.3 and the remark thereafter)

$$P'Q' = |St(v, d\mathscr{E}')| = C_{\frac{1}{2}}(S * \sum),$$
$$P' - P' \cap Q' = (P' \cap \partial E) \times [\frac{1}{2}, 1], \quad \text{and}$$
$$Q' - P' \cap Q' = (Q' \cap \partial E) \times [\frac{1}{2}, 1].$$

And in this case (1) and (2) are obvious. □

Proposition 8.5.6. *Suppose A and B are spheres of dimensions p and q respectively, contained in the interior of a* $(p + q)$*-manifold C and that A and B intersect at a single point x transversally in C. Then there are regular neighbourhoods N and T of A and B in C, such that*

(1) *$N \cap T$ is a submanifold of both N and T*

(2) *$N - (N \cap T)$ and $T - (N \cap T)$ are both $(p + q)$-cells.*

Proof. Let F be the nice neighbourhood of x in C given by 8.5.4 i.e. there is a polyhedral equivalence $f : F \approx E = S * \sum * v$ where S is a $(p-1)$-sphere and $\sum$ a $(q-1)$-sphere, such that $f(x) = v$, $f(A \cap F) = S * v$, and $f(B \cap F) = \sum * v$. Then $(\overline{A - F})$ is a $(p - 1)$-cell and $(\overline{B - F})$ is a $(q - 1)$-cell. Let $\mathscr{S}_1$ and $\mathscr{E}_1$ be triangulations of F and E such that f is simplicial with reference to $\mathscr{S}_1$ and $\mathscr{E}_1$. We can assume $\mathscr{E}_1$ contains full subpresentations covering $S * v$ and $\sum * v$. Now some refinement $\mathscr{S}$ of $\mathscr{S}_1$ can be extended to a neighbourhood of $A \cup B$, denote it by

$\mathscr{S}'$, it can be supposed that $\mathscr{S}'$ contains full subpresentations $\mathfrak{a}$, $\mathscr{B}$ covering A, B respectively. Let η be a centering of $\mathscr{S}'$. Denote by $\mathscr{E}$ the **221**
triangulation of E corresponding to $\mathscr{S}$ by f, and by $\mathfrak{b}$ the centering of $\mathscr{E}$ corresponding to $\eta|\mathscr{S}$. Choose $N = |N_{\mathscr{S}'}(\mathfrak{a})|$ and $T = |N_{\mathscr{S}'}(\mathscr{B})|$; and let P, Q be as in 8.5.5. If $P_1 = f^{-1}(P)$, $Q_1 = f^{-1}(Q)$, then $P_1 = (P_1 \cap Q_1) \searrow P_1 \cap \partial F$ and $Q_1 - (P_1 \cap Q_1) \searrow Q_1 \cap \partial F$. Clearly $P_1 \subset N$, $Q_1 \subset T$ are submanifolds and $N \cap T = P_1 \cap Q_1$. Thus $N \cap T$ is a submanifold of both N and T. $N - (N \cap T) = N - (P_1 \cap Q_1) = (N - P_1) \cup (P_1 - (P_1 \cap Q_1))$ collapses to $(N - P_1)$ since $P_1 - (P_1 \cap Q_1)$ collapses $P_1 \cap \partial F \subset (N - P_1)$. But $N - P_1$ is a regular neighbourhood of $\overline{A - F}$ in $C - F$ which is a $(p-1)$-cell. Thus $N - (N \cap T) \searrow N - P_1 \searrow \overline{A - F}$ which is collapsible. Thus $N - (N \cap T)$ is a collapsible $(p+q)$-manifold, hence a $(p+q)$-cell. Similarly $T - (N \cap T)$ is a $(p+q)$-cell. □

Definition 8.5.7. Let $\mathscr{H} = (A_{-1}, \ldots, A_n)$ be a handle presentation of a relative n-manifold (M, X). Let $\mathfrak{h}$ be a k-handle and $\mathfrak{K}$ be a $(k+1)$-handle of $\mathscr{H}$. We say that $(\mathfrak{h}, \mathfrak{K})$ *can be nearly cancelled* if there is a transverse sphere S of $\mathfrak{h}$ and an attaching sphere $\sum$ of $\mathfrak{K}$ which intersect a single point transversally in $\partial(A_k, X)$.

Proposition 8.5.8. *Suppose $\mathscr{H}$ is a handle presentation of relative n-manifold (M, X), $\mathfrak{h}$ a k-handle and $\mathfrak{K}$ a $(k+1)$-handle in $\mathscr{H}$. If $\mathfrak{h}$ and $\mathfrak{K}$ can be nearly cancelled, then there is a polyhedral equivalence $f : A_k \to A_k$ isotopic to the identity leaving X fixed such that, in $\mathscr{H}_f$ the handles $\mathfrak{h}'$ and $\mathfrak{K}'(= \mathfrak{K})$ corresponding to $\mathfrak{h}$ and $\mathfrak{K}$ can be cancelled.*

Proof. Follows from 8.4.3 and 8.5.6. □

8.6 Insertion of cancelling pairs of handles

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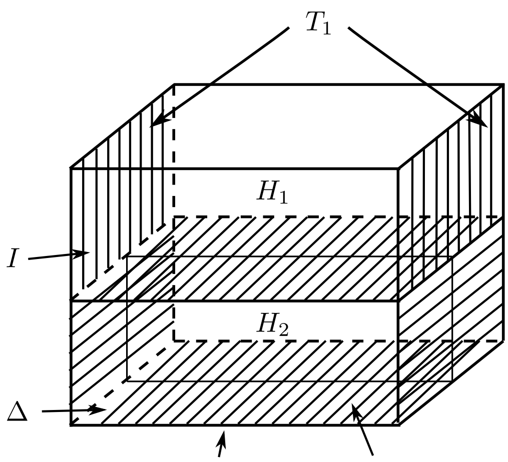
In this section we discuss the insertion of cancelling pairs of handles and two applications which are used in the following sections. First we form a standard trivial pair as follows:



Let D be a k-cell, $I = [0,1]$ and Δ an $(n-k-1)$-cell. Then $E = D \times I \times \Delta$ is an n-cell. Let

$$H_1 = D \times [\frac{1}{2}, 1] \times \Delta$$

$$T_1 = \partial D \times [\frac{1}{2}, 1] \times \Delta$$

Clearly $\mathfrak{h} = (H_1, T_1)$ is a handle of index k. Next, let

$$\begin{aligned} H_2 &= D \times [0, \frac{1}{2}] \times \Delta \\ T_2 &= \partial\{D \times [0, \frac{1}{2}]\} \times D \\ &= \{(D \times 0) \cup (D \times \frac{1}{2}) \cup (\partial D \times [0, \frac{1}{2}])\} \times \Delta. \end{aligned}$$

Clearly, $\mathfrak{K} = (H_2, T_2)$ is a handle of index $(k+1)$. Finally, let F denote $(D \times 0 \times \Delta) \cup (\partial D \times I \times \Delta)$. $(D \times 0 \times \Delta) \cap (\partial D \times I \times \Delta) = \partial D \times 0 \times \Delta$
223 is an $(n-2)$-manifold, hence F is an $(n-1)$-manifold. Moreover F is collapsible, hence it is an $(n-1)$-cell.

Now, let $\mathscr{H}$ be a handle presentation of a relative n-manifold (M, X); we take an $(n-1)$-cell F' in $\partial(A_k, X)$ away from the k-and $(k+1)$-handles. That is F' is in the common portion of $\partial(A_{k-1}, X)$, $\partial(A_k, X)$ and

$\partial(A_{k+1}, X)$ and clearly it is possible to choose such an F' if $(k+1) < n$, that is $k \leq n-2$. Now we take some equivalence $\alpha : F \approx F'$ and attach E to A_k by α. Denote the result by $A_k \cup E$. Since E is an n-cell meeting $\partial(A_k, X)$ in an $(n-1)$-cell F', there is an equivalence $f : A_k \approx A_k \cup E$ leaving X fixed. Then we get a new handles presentation $(B_{-1}, \ldots, B_n)$ of (M, X) as follows:

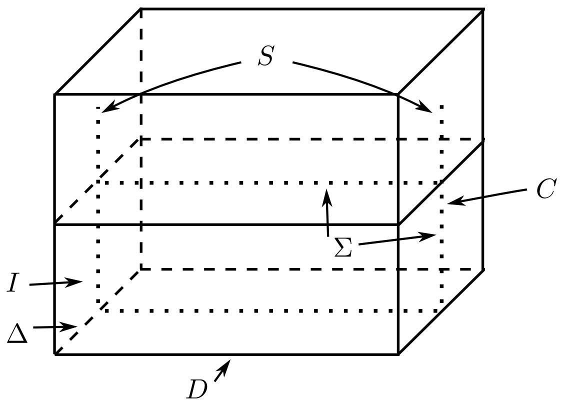
$$\begin{aligned} B_i &= f^{-1}(A_i), \quad \text{for} \quad i < k, \\ B_k &= f^{-1}(A_k + \mathfrak{h}) \\ B_i &= A_i \quad \text{for} \quad i > k. \end{aligned}$$

Next we consider the problem of attaching a cancelling pair of k- and $(k+1)$-handles $(\mathfrak{h}, \mathfrak{R})$ to A_k, with $\mathfrak{h}$ having a prescribed attaching sphere. We recall from Chapter 7 (7.2) that a sphere in the interior of a PL-manifold N is unknotted (by definition) if it bounds a k-cell. In such a case it bounds an unknotted cell (again in the sense of 7.2). If S and S' are two unknotted k-spheres in the same component of $N - \partial N$, then there is an isotopy h_t of N leaving N fixed such that $h_1(S) = S'$. Similarly if D and D' are two unknotted k-cells in the same component of $N - \partial N$, there is an isotopy of N taking D onto D'. Similar remarks apply in the case of relative manifolds also. **224**

Now consider just A_k, let F' be any $(n-1)$-cell in $\partial(A_k, X)$ and form $A_k \cup E$ by an equivalence $\beta : F \approx F'$. Consider $S = \partial D \times \alpha \times e$, where $\frac{1}{2} < \alpha < 1$ and $e \in \Delta - \partial\Delta$. S is an attaching sphere of $\mathfrak{h}$, and $\sum = \partial\{D \times [0, \frac{1}{2}]\} \times e$

$$= \{(D \times 0) \cup (D \times \frac{1}{2}) \cup \partial D \times [0, \frac{1}{2}]\} \times e$$

is an attaching sphere of $\mathfrak{R}$.



And $\{D \times 0 \cup \partial D \times [0, \alpha]\} \times e = C$, say, is a k-cell bounding $S = \partial D \times \alpha \times e$, an attaching sphere of $\mathfrak{h}$. Moreover

$$
\begin{aligned}
C \cap \sum &= \{D \times 0 \cup \partial D \times [0, \frac{1}{2}]\} \times e \\
&= C - \partial D \times [\frac{1}{2}, \alpha] \times e \\
&= C - (\text{a regular neighbourhood of } S \text{ in } C).
\end{aligned}
$$

Finally C is unknotted in F.

The result of all this is, if A is a $(k-1)$-sphere bounding an unknotted
225 k-cell B in $\partial(A_k, X)$, then we can attach a cancelling pair of k- and $(k+1)$-handles $(\mathfrak{h}, \mathfrak{K})$ such that A is an attaching sphere of $\mathfrak{h}$ and an attaching sphere of $\mathfrak{K}$ intersects $\partial(A_k, X)$ in B - (a given a regular neighbourhood of A in B). This can also be seen as follows:

Let L be an $(n-1)$-cell, A a $(k-1)$-sphere in L bounding an unknotted k-cell B in the interior of L. Let M be an n-cell containing L in its boundary. We may join A and B to an interior point v of M and take second derived neighbourhoods. Let H be a second derived neighbourhood of $A * v$ and K be the closure of [second derived neighbourhood of $B * v - H$]. Then $(H, H \cap L)$ is a k-handle, and $(K, (K \cap H) \cup (K \cap L))$ is a $(k+1)$-handle. The k-handle has A as an attaching sphere, and an attaching sphere of the $(k+1)$-handle intersects L in (B - a regular neighbourhood of A in B). Thus we have,

8.6.1 Let $\mathscr{H} = (A_{-1}, \ldots, A_n)$ be a handle presentation of a relative n-manifold (M, X) and let $S \subset \partial(A_k, X)$ be a $(k-1)$-sphere which bounds an unknotted cell T in $\partial(A_k, X)$. Then there are a k-handle $\mathfrak{h}$ and a $(k+1)$-handle $\mathfrak{K}$, such that

(1) S is an attaching sphere of $\mathfrak{h}$

(2) There is an attaching sphere $\sum$ of $\mathfrak{K}$ with $\sum \cap A_k$ very closed to T, that is $\sum \cap A_k$ can be assumed to be (T - a prescribed regular neighbourhood of S in T).

(3) $((A_k, X) + \mathfrak{h}) + \mathfrak{K}$ exists and is polyhedrally equivalent to (A_k, X) by an equivalent which is identity outside a given neighbourhood **226**
of T in A_k.

If S is in $\partial(A_{k-1}, X) \cap \partial(A_k, X)$ we can choose $\mathfrak{h}$ to have its attaching tube in $\partial(A_{k-1}, X)$, so that there is an obvious handle presentation of $((A_k + \mathfrak{h}) + \mathfrak{K}, X)$. We give below two applications of this construction.

8.6.2 Trading handles. Let $\mathscr{H} = (A_{-1}, \ldots, A_n)$ be a handle presentation of a relative n-manifold (M, X). Let p_i be the number of i-handles in $\mathscr{H}$. Suppose that there is a $(k-1)$-handle ℓ $(2 \leq k \leq n-1)$ with a transverse sphere $\sum$, and that there is a $(k-1)$-sphere S in $\partial(A_{k-1}, X) \cap \partial(A_k, X)$ such that (1) S is unknotted in $\partial(A_k, X)$, (2) S intersects $\sum$ transversally at exactly one point in $\partial(A_{k-1}, X)$. Then there is a procedure by which we can obtain another handle presentation $\mathscr{H}'$ of (M, X), such that (a) for $i \neq k-1$ or $k+1$, the number of i-handles in $\mathscr{H}$ is equal to the number of i-handles in $\mathscr{H}'$, (b) the number of $(k-1)$-handles in $\mathscr{H}'$ is $p_{(k-1)} - 1$ (c) the number of $(k+1)$-handles in $\mathscr{H}'$ is $p_{(k+1)} + 1$. This is done as follows:

First consider only A_k. Applying 8.6.1, we can add to A_k a cancelling pair of k- and $(k+1)$-handles $(\mathfrak{h}, \mathfrak{K})$ such that S is an attaching sphere of $\mathfrak{h}$, and the attaching tube of S is in $\partial(A_{k-1}, X)$. Write $(A_k + \mathfrak{h}) + \mathfrak{K} = B$. Then the relative manifold (B, X) has the obvious handle presentation $\mathscr{K}' = (B_{-1}, \ldots, B_{k+1})$ where

$$B_i = A_i, \quad \text{for} \quad i \leq k-1$$

$$B_k = A_k + \mathfrak{h}$$
$$B_{k+1} = (A_k + \mathfrak{h}) + \Re = B.$$

227 In $\Re'$, the handles ℓ and $\mathfrak{h}$ can be nearly cancelled. Hence for some equivalence f of A_{k-1}, isotopic to identity and leaving X fixed, in $\mathscr{K}'_f$, the handles ℓ' and $\mathfrak{h}'(=\mathfrak{h})$ corresponding to ℓ and $\mathfrak{h}$ can be cancelled. Let $\mathscr{K}'' = \mathscr{K}'_f - (\mathfrak{h}', \ell')$. $\mathscr{K}''$ is a handle presentation of (B, X); the number of i-handle in $\mathscr{K}''$ for $i \leq k-1$ is p_i, the number of $(k-1)$-handles is $\mathscr{K}''$ is $p_{k-1} - 1$, the number of k-handles is p_k and there is one $(k+1)$-handle. Also there is an equivalence $\alpha : A_k \to B$ which can be assumed to be identity near X. Thus we can pull back $\mathscr{K}''$ to a handle presentation $\mathscr{K}$ of (A_k, X) by α^{-1}.

Now, we would like to add the $(\geq k+1)$-handles of $\mathscr{H}$ to $\mathscr{K}$ to get a new handle presentation of (M, X). But it may happen that the attaching tubes of the $(k+1)$-handles of $\mathscr{H}$ intersect the transverse tube of $\alpha^{-1}(\Re)$ which is in $\partial(A_k, X)$. However, we can adopt the procedure of 8.4.5, to get the desired type of handle presentations as follows:

Let $\Re_1^{(k+1)}$, $\Re_2^{(k+1)}$, $\Re_{p_{(k+1)}}^{(k+1)}$ be the $(k+1)$-handles of $\mathscr{H}$, with attaching tubes $T_1, T_2, \ldots, T_{p_{(k+1)}}$ respectively. Choose some attaching spheres $S_1, \ldots, S_{p_{(k+1)}}$ of these handles, and then a transverse sphere $\sum$, of $\alpha^{-1}(\Re)$ avoiding $S_1, \ldots, S_{p_{(k+1)}}$. This is done in the same way as
228 in 8.4.5, using the product structure of the transverse tube of $\alpha^{-1}(\Re)$ as $D^{k+1} \times \Delta^{n-k-1}$ and noticing that the S_i are now k-dimensional. Then choose a regular neighbourhood N_1 of $\sum_1$ which does not intersect the S_i's and do a modification of type 8.4.1 so that, for some g, in $\mathscr{K}_g$ the handle $\Re'$ corresponding $\alpha^{-1}(\Re)$ has N_1 as its transverse tube. Now choose regular neighbourhoods T'_i of S_i in $\partial(A_k, X)$ such that $T'_i \cap N_1 = \emptyset$ for all i and $T'_i \cap T'_j = \emptyset$ for all i, j, $i \neq j$. There is an equivalence β of A_k isotopic to the identity leaving X fixed such that $\beta(T_i) = T'_i$ for all i. Now attach the handles $\Re_i^{(k+1)}$ to A_k not by the inclusion of T_i but by $\beta|T_i$. Then we obtain a relative n-manifold say (C, X) and a genuine handle presentation say $\mathscr{K}_1$ of (C, X). Moreover the equivalence β of A_k can be extended to an equivalence β_{k+1} of A_{k+1} with C. Now pull back $\mathscr{K}_1$ to A_{k+1} by $(\beta_{k+1})^{-1}$. In the handle presentation $(\beta_{k+1})^{-1}(\mathscr{K}_1)$ of A_{k+1} there are handles only upto index $(k+1)$; so that the handle of index

$\geq k+2$ of $\mathscr{H}$ can be added as they are to get a handle presentation of (M, X) of the derived type.

8.6.3 The second application is concerning the maps in the homotopy groups: $\pi_k(A_k, A_{k-1}) \xrightarrow{b_k} \pi_{k-1}(A_{k-1}, A_{k-2})$. It will be seen later that under suitable assumptions, these are free $Z\pi$-modules with more or less well defined bases. The problem is to find handle presentations for which the matrices of b_k's with reference to preferred bases will be in some **229**
convenient form (8.9). Here we describe an application of 8.6.1 which is useful for this purpose.

Let N be a PL n-manifild, and assume that ∂N is connected. Let $\mathfrak{h}_1$, $\mathfrak{h}_2$ be two k-handles ($2 \leq k \leq n-2$) so that $n \geq 4$) attached to N. If we choose a cell in ∂N intersecting the handles as "base point", any attaching sphere of $\mathfrak{h}_1(\mathfrak{h}_2)$ determines a well defined element in $\pi_{k-1}(\partial N)$. Let the elements in $\pi_{k-1}(\partial N)$ determined by $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be α_1 and α_2. Let θ be an element of $\pi_1(\partial N)$. Imagine that the handles are in the form $\mathfrak{h}_i = (D_i \times \Delta_i, \partial D_i \times \Delta_i)$, D_i a k-cell, Δ_i an $(n-k)$-cell $i = 1, 2$. Let $p_i \in \partial\Delta_i$. Then, we have surface cores $C_i = D_i \times p_i$ of $\mathfrak{h}_i$, and representatives $S_i = \partial D_i \times p_i$ of α_i. Let P be a path between a point of S_1 and a point of S_2 in ∂N representing θ. Since $n \geq 4$, we can assume that P is an embedded arc, and since $k \leq n-2$, that it meets each S_i at exactly one point. Now P appears also as an arc joining C_1 and C_2. Thicken P, so that we have an $(n-1)$-cell Q which intersects C_1 and C_2 in $(k-1)$-dimensional arcs E_1 and E_2 with $E_i = \partial C_i \cap \partial Q$. We can be careful enough to arrange for E_i to be unknotted in ∂Q, so that there is a k-cell $F \subset Q$ with $\partial F \cap \partial Q = E_1 \cup E_2$.

The composite object $C_1 \cup F \cup C_2$ is now a k-cell with boundary $(S_1 - E_1) \cup [\partial F - (E_1 \cup E_2)] \cup (S_2 - E_2)$, which represents in $\pi_{k-1}(\partial N)$ the element $\alpha_1 \pm \theta\alpha_2$. The sign depends on F, and we can choose F so as to have the prescribed sign (see Chapter 7). Moreover we can assume that $C_1 \cup F \cup C_2$ is unknotted in $\partial((N + \mathfrak{h}_1) + \mathfrak{h}_2))$. Stretch $C_1 \cup F \cup C_2$ a **230**
little to another unknotted k-cell T so that $S = \partial T \subset (\partial N$ - union of the attaching tubes of $\mathfrak{h}_1$ and $\mathfrak{h}_2$). That is, we have a $(k-1)$-sphere S in ∂N representing $\alpha_1 + \epsilon\theta\alpha_2$ ($\epsilon = \pm 1$, prescribed) and bounding an unknotted cell T in $\partial(N + \mathfrak{h}_1 + \mathfrak{h}_2)$ and S is away from $\mathfrak{h}_1$ and $\mathfrak{h}_2$. We now add a

cancelling pair of k- and $(k+1)$-handles $\mathfrak{h}$ and $\mathfrak{K}$, so that an attaching sphere of $\mathfrak{h}$ is S and an attaching sphere of $\mathfrak{K}$ intersects $\partial((N+\mathfrak{h}_1+\mathfrak{h}_2)$ along $C_1 \cup F \cup C_2$.

Now,

$$N + \mathfrak{h}_1 + \mathfrak{h}_2 \approx ((N + \mathfrak{h}_1 + \mathfrak{h}_2) + \mathfrak{h}) + \mathfrak{K}$$

But then h_1 and $\mathfrak{K}$ nearly cancel, since attaching sphere of $\mathfrak{K}$ intersects a transverse sphere of $\mathfrak{h}_1$ exactly as C_1 does, that is, at one point, transversally. So that, after an isotopy we can find a $(k+1)$-handle $\mathfrak{K}'$ such that $\mathfrak{h}_1$ and $\mathfrak{K}'$ actually cancel. Thus

$$\begin{aligned}(N + \mathfrak{h}_1 + \mathfrak{h}_2) + \mathfrak{h} + \mathfrak{K} &\approx (N + \mathfrak{h}_1 + \mathfrak{h}_2) + \mathfrak{h} + \mathfrak{K}' \\ &\approx (N + \mathfrak{h}_2 + \mathfrak{h}).\end{aligned}$$

We have proved,

Proposition 8.6.4. *Let N be a PL n-manifold, with connected boundary ∂N; $n \geq 4$. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two handles attached to N, and α_1, α_2 be the elements in $\pi_{k-1}(\partial N)$ given by $\mathfrak{h}_1$ and $\mathfrak{h}_2$; and θ be an element of $\pi_1(\partial N)$. Then there exists a handle $\mathfrak{h}$ which can be attached to N, with its attaching tube away from $\mathfrak{h}_1$ and $\mathfrak{h}_2$ so that $N+\mathfrak{h}_1+\mathfrak{h}_2 \approx N+\mathfrak{h}+\mathfrak{h}_2$, and the element of $\pi_{k-1}(\partial N)$ represented by $\mathfrak{h}$ is $\alpha_1 \pm \theta\alpha_2$, sign prescribed.*

231 **Remark 1.** Some details, such as thickening of P, choosing certains cells so as to be unknotted; are left out. These are easy to verify using our definition of unknotted cells and choosing regular neighbourhoods in the appropriate manifolds. There is another point to check: that the homotopy groups can be defined with cells as 'base points', so that we can get away without spoiling the embeddings (of attaching spheres in appropriate dimensions), when forming sums in the homotopy groups or the action of an element of the fundamental group.

Remark 2. In 8.6.4, instead of the whole of ∂N, we may as well take a connected $(n-1)$-manifold N' in ∂N and do every thing in its interior of course, now $\alpha_1, \alpha_2 \in \pi_{r-1}(N')$ and $\theta \in \pi_1(N')$.

Remark 3. The proof can also be completed by observing that S and S_1 differ by cellular moves in $(N + \mathfrak{h}_2)$.

8.7 Elimination of 0 - and 1-handles

The first thing to do is to remove all handles of index 0, and 1 to attain a stage where $\pi_1(A_k) \approx \pi_1(M)$. At this point we can interpret $\pi_i(A_i, A_{i-1})$ and so on as homology groups in universal covering spaces and this helps things along.

Proposition 8.7.1. *Let (M, X) be a relative manifold, M connected, $X \neq \emptyset$, and $\mathscr{H}$ a handle presentation of (M, X). Then all the 0-handles of $\mathscr{H}$ can be eliminated by cancelling pairs of 0- and 1-handles of $\mathscr{H}$ to obtain a handle presentation of (M, X) free of 0-handles.*

Proof. A 0-handle $\mathfrak{h} = (H, \emptyset)$ and a 1-handle $\mathfrak{K} = (K, T)$ cancel if only if **232**
the attaching sphere $\sum$ of $\mathfrak{K}$ intersects $\mathfrak{h}$ in a single point; for the attaching tube T of $\mathfrak{K}$ consists of two disjoint $(n-1)$-cells, and the transverse tube of $\mathfrak{h}$ is ∂H, and so what we need is for exactly one of the $(n-1)$-cells of T to be in ∂H. So all the 0-handles of $\mathscr{H}$ which are connected to A_{-1} ($\neq \emptyset$, since $X \neq \emptyset$) by means of 0- and 1-handles can be eliminated. But every 0-handle must be one such; for if ℓ is a 0-handle of $\mathscr{H}$ which is not connected to A_{-1} by 0- and 1-handles, then ℓ together with all the 0- and 1-handles connected to it will form a component of A_1 which is totally disjoint from A_{-1}. Thus A_1 has at least two components, and so, since $\pi_0(A_1) \to \pi_0(M)$ is an isomorphism, we have a contradiction to the assumption that M is connected. □

For the next stage, we need a lemma:

Lemma 8.7.1. *A null homotopic 1-sphere in the interior of a PL-manifold M of dimension ≥ 4 is unknotted.*

Proof. Let S be a null homotopic 1-sphere in the interior of M. We have to show that S bounds a 2-cell in M. Let D be a 2-cell, and α an equivalence of ∂D with S. Since S is null homotopic α extends to D. Approximate α by a map β in general position such that $\beta|\partial D = \alpha|\partial D$, and $\beta(D) \subset \operatorname{int} M$. The singular set $S_2(\beta)$ of β consists of finite number of points and $S_3(\beta)$ etc. are all empty. So we can partion $S_2(\beta)$ into two sets $\{p_1, \ldots, p_m\}$, $\{q_1, \ldots, q_m\}$ such that $\beta(p_i) = \beta(q_i)$, $1 \leq i \leq m$ and there are no other identifications. Choose some point p on ∂D and

join $\{p, p_1, \ldots, p_m\}$ by an embedded are γ which does not meet any of 233
the q_i's. Let N be a regular neighbourhood of γ in D, which does not contain any of the q_i's. N is a 2-cell.

Let $N \cap D = \partial N \cap \partial D = L$, $\partial N - L = K$, and $D - N = D'$. Since L is a 1-cell, K is also 1-cell, and D' is a 2-cell. And $\beta|N$ as well as $\beta|D'$ are embeddings. So $\beta(\partial D')$ is unknotted in M. But by 7.1.6, there is an isotopy carrying $\beta(L)$ to $\beta(K)$ and leaving $\beta(\partial D' - K)$ fixed, that is, the isotopy carries S onto $\beta(\partial D')$. Hence S is also unknotted. □

Remarks:

(1) The same proof works in the case of a null homotopic n-sphere in the interior of $a \geq (2n + 2)$-dimensional manifold.

(2) The corresponding lemma is true in the case of relative manifolds also.

(3) If S is in ∂M, then the result is not known. It is conjectured by Zeeman, that the lemma in this case is in general false (e.g. in the case of contractible 4 dimensional manifolds of Mazur).

Proposition 8.7.2. *Let $\mathscr{H} = (A_{-1}, \ldots, A_n)$ be a handle presentation without* 0*-handles of relative n-manifold* (M, X) *and let* $\pi_1(M, A_{-1}) = 0$. *Then by admissible changes involving the insertion of* 2*- and* 3*-handles and the cancelling of* 1*- and* 2*-handles, we can obtain from $\mathscr{H}$ a handle presentation of* (M, X) *without* 0*- or* 1*-handles, provided $n \geq 5$.*

Proof. Let $\mathfrak{h}$ be a 1-handle of $\mathscr{H}$. By 8.4.5, we can assume that there is a surface core of $\mathfrak{h}$ in $\partial(A_2, X)$.

Because $\pi_1(M, A_{-1}) = 0$, then $\pi_1(A_2, A_{-1}) = 0$ (from the homotopy
234 exact sequence of the triple (M, A_2, A_{-1}) and so C is homotopic leaving its end points fixed to a path in A_{-1}. $\partial(A_{-1}, X) \subset A_{-1}$ is a homotopy equivalence (we are confining ourselves to the special case after 8.3). So we get a map, where D is a 2-cell

$$f : D \to A_2 - \operatorname{int} A_{-1}$$

with $\partial D \supset C$, such that $f(\partial D - C) \subset \partial(A_{-1}, X)$ and $f|C = \mathrm{Id}_C$.

Now in the ($\geq$ 4)-manifold $\partial(A_{-1}, X)$ the removal of the attaching tubes of the 1-handles does not disturb any homotopy of dimension ≤ 2, so that, we can arrange for

$$\begin{aligned} f(\partial D - C) &\subset \partial(A_{-1}, X)\text{- (attaching tubes of 1-handels)} \\ &\subset \partial(A_1, X) \quad \text{(since } A_{-1} = A_0\text{)}. \end{aligned}$$

Likewise in $\partial(A_1, X)$, the removal of the attaching tubes of 2-handles can be ignored as far as one-dimensional things go, so that we can assume

$$f(\partial D - C) \subset \partial(A_2, X),$$

and that $f|\partial D$ is an embedding. Also, we can arrange $f(\partial D)$ to intersect $\mathfrak{h}$ precisely along C.

Finally, then we have

$$\begin{aligned} & f : D \to A_2 \\ \text{with} \quad & f(\partial D) \subset \partial(A_2, X) \cap \partial(A_1, X) \\ & f|C = \mathrm{Id}_C, \quad \text{and this is the only place where} \end{aligned}$$

$f(\partial D)$ intersects $\mathfrak{h}$. Hence $f(\partial D)$ intersects atransverse sphere of $\mathfrak{h}$ at eactly one point transversally.

Now, upto homotopy, A_2 is obtained from $\partial(A_2, X)$ by attaching cells of dimensions $(n-2)$ and $(n-1)$ [cf. duality 8.8]. Since $(n-$ **235**
$2) \geq 3$, $\pi_2(A_2, \partial(A_2, X)) = 0$. Thus the map f can be deformed into $\partial(A_2, X)$ leaving $f|\partial D$ fixed. Thus the 1-sphere $f(\partial D)$ is null homotopic in $\partial(A_2, X)$, hence by Lemma 8.7.1 it is unknotted in $\partial(A_2, X)$. Now we can apply 8.6.3 to trade $\mathfrak{h}$ for a 3-handle. We can apply this procedure successively until all the 1-handles are eliminated. Since in this procedure, only the number of 1-handles and 3-handles is changed, in the final handle presentation of (M, X) there will be no 0-handles either. $\square$

Remark. If (M, X) is ℓ-connected and $2\ell + 3 \leq n$, we can adopt the above procedure to get a handle presentation of (M, X) without handles of index $\leq \ell$.

8.8 Dualisation

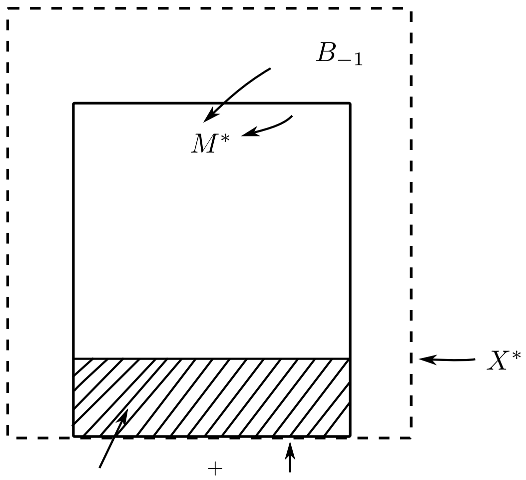
In this section, we discuss a sort of dualization, which is useful in getting rid of the very high dimensional handles.

Let (M, X) be a relative n-manifold (remember that we are dealing with the special cae; X and $(n-1)$-submanifold of ∂M), and let $\mathscr{H}$ be a handle presentation of (M, X). Consider the manifold M^+ obtained from M by attaching a collar over $\overline{\partial(M, X)}$ (= $\partial M - X$ by the notation of 8.5).

$$M^+ = \{M \cup (\partial M - X) \times [0, 1]\}$$

identifying x with $(x, 0)$ for $x \in \partial M - X$. Let

$$\begin{aligned} M^* &= M^+ - A_{-1} \\ X^* &= \{(\partial M - X) \times 1\} \cup \{\partial(\partial M - X) \times [0, 1]\} \\ \text{and} \quad X^+ &= (\partial M - X) \times 1. \end{aligned}$$



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We consider (M^*, X^*) as a dual of (M, X). Now $\mathscr{H}$ gives rise to a handle presentation $\mathscr{H}^* = (B_{-1}, \ldots, B_n)$ of (M^*, X^*) as follows:

$$B_{-1} = (\partial M - X) \times [0, 1]$$

$$\begin{aligned} B_k &= M^+ - A_{n-k-1} \\ &= B_{k-1} + \mathfrak{h}_1^* + \cdots + \mathfrak{h}_{p(n-k)}^*. \end{aligned}$$

Where $\mathfrak{h}_1, \ldots, \mathfrak{h}_{p(n-k)}$ are the $(n-k)$-handles of $\mathscr{H}$. This $\mathscr{H}^*$, we will call the dual of $\mathscr{H}$. The number of k-handles in $\mathscr{H}$ is equal to the number of $(n-k)$-handles in $\mathscr{H}^*$.

Now,

$$\begin{aligned} &\partial M^* = X^* \cup \overline{\partial(A_{-1}, X)} \\ \text{so that}\quad &\partial(M^*, X^*) = \partial(A_{-1}, X). \end{aligned}$$

Since A_{-1} is a collar over $\overline{\partial(A_{-1}, X)}$, this shows that (M, X) is a dual of (M^*, X^*); and with this choice of the dual pair $\mathscr{H}$ is the dual of $\mathscr{H}^*$. **237**

Given any handle presentation $\mathscr{K} = (C_{-1}, \ldots, C_n)$ of (M^*, X^*) with $C_{-1} = B_{-1}$, then we obviously get a handle presentation $\mathscr{K}^*$ of (M, X). Even if $C_{-1} \neq B_{-1}$, we can get a handle presentation of (M, X) whose number of k-handles is equal to the number of $(n-k)$-handles of $\mathscr{K}$ as follows:

Let $X^+ = \overline{\partial(M, X)} \times 1$. In M^+, $C_{-1} \searrow X^*$ and $X^* \searrow X$, (both) homogeneously. Since C_{-1} is a collar over X^*; by using the theorems about cells in spheres and cells in cells, we see that C_{-1} is bicollared in M^+. Moreover C_{-1} is a neighbourhood of X^+ in M^+. Hence by the regular neighbourhood theorem, C_{-1} is a regular neighbourhood of X^+ in M^+. But B_{-1} is also a regular neighbourhood of X^+ in M^+. Therefore, there is an equivalence f of M^+, fixing X^+, with $f(C_{-1}) = B_{-1}$. Since $f(\partial M^+) = \partial M^+$ and $C_{-1} \cap \partial M^+ = B_{-1} \cap \partial M^+ = X^*$, f maps X^* onto itself, and as $\partial M^+ = X \cup X^*$, f has to map X onto itself. Now the desired handle presentation of (M, X) is given by

$$\begin{aligned} D_{-1} &= f(A_{-1}) \quad (\text{since } A_{-1} \searrow X,\ f(A_{-1}) \searrow f(X) = X) \\ D_k &= M^+ - f(C_{n-k-1}) \\ &= D_{k-1} + (f(\mathfrak{K}_1))^* + \cdots + (f(\mathfrak{K}_{p(n-k)}))^* \end{aligned}$$

where $\mathfrak{K}_1, \ldots, \mathfrak{K}_{p(n-k)}$ are the $(n-k)$-handles of $\mathscr{K}$.

Thus

8.8.1 If there is a handle presentation of (M^*, X^*) without handles of 238
index $\leq n - \ell$, then there is a handle presentation of (M, X) without handles of index $\geq \ell$. This gives:

8.8.2 Theorems A and B imply Theorem C.

Since $X \hookrightarrow M$ is a homotopy equivalence and $\pi_1(M) \approx \pi_1(\partial(M, X))$, using duality in the universal covering spaces, that is $H_i(M^*, X^*) \approx H^{n-i}(M, X) = 0$, we see that $X^* \hookrightarrow M^*$ is also a homotopy equivalence. If $n \geq 6$, then we can find a handle presentation of (M^*, X^*) without handles of index $\leq 6 - 4 = 2$ by Theorem A. Hence we can obtain handle presentation $\mathscr{H}$ of (M, X) without handles of index $\geq n - 2$, that is, with handles of index $\leq n - 3$ only. But then, by Theorem B, as $\tau(M, X) = 0$, we can get from $\mathscr{H}$ a handle presentation of (M, X) without any handles, that is $M \searrow X$.

8.8.3 If $n = 5$, and (M, X) is a h-cobordism, then there is a handle presentation of (M, X) with only 2- and 3-handles.

Ex. 8.8.4. A (compact) contractible PL 2-manifold is a 2-cell.

8.9 Algebraic Description

We have already remarked (8.3.7) that there is a certain algebraic structure associated to a handle presentation $\mathscr{H} = (A_{-1}, \ldots, A_n)$ of a relative n-manifold (M, X). We suppose now that (M, X) is a special case, and that there are no 0- or 1-handles in $\mathscr{H}(A_{-1} = A_0 = A_1)$. Also $n \geq 3$ and $\pi_1(X) \to \pi_1(M)$ is an isomorphism. This we will call Hypothesis

8.9.1 In this case, the maps

$$\pi_1(X) \to \pi_1(A_{-1}) \to \ldots \to \pi_1(A_n)$$

are all isomorphisms. The reason $\pi_1(A_1) \to \pi_1(A_2)$ is an isomorphism
239 is that $\pi_1(X) \to \pi_1(M)$ is an isomorphism and $\pi_1(A_1) \to \pi_1(A_2)$ is a
surjection. We identity all these groups and call it π.

Now, the groups

$$C_i = \pi_i(A_i, A_{i-1})$$

are identified with $H_i(\tilde{A}_i, \tilde{A}_{i-1})$. They are free modules over $Z\pi$ with bases corresponding to handles. $\mathscr{H}\{\mathfrak{h}_1^{(i)}, \ldots, \mathfrak{h}_{p_i}^{(i)}\}$ are the i-handles, the basis of C_i is denoted by $\{[\mathfrak{h}_1^{(i)}], \ldots, [\mathfrak{h}_{p_i}^{(i)}]\}$ and the elements of this basis are well defined upto multiplying by elements $\pm\pi$.

If $f : A_k \to A_k$ is a polyhedral equivalence isotopic to the identity, it is easily seen that the algebraic structures already described for $\mathscr{H}$ and $\mathscr{H}_f$ may be identified.

In addition, we have a map

$$\partial_k : C_k \to C_{k-1}$$

which is the boundary map of the triple (A_k, A_{k-1}, A_{k-2}). This is also unchanged by changing $\mathscr{H}$ to $\mathscr{H}_f$.

If there are no handles of index $\leq k-2$ and $\pi_{k-1}(M, X) = 0$, we see:

First, $\pi_{k-1}(A_k, A_{k-2}) = 0$, and hence from the exact sequence of the triple (A_k, A_{k-1}, A_{k-2}) the map $\partial_k : C_k \to C_{k-1}$ is surjective.

Dually, if there are no handles of index $> k$ and $\pi_k(X, X) = 0$, we have

$$\partial_k : C_k \to C_{k-1} \quad \text{to be injective.}$$

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Now, the boundary map ∂_k plus the bases of C_k and C_{k-1} determine a matrix B_k in the usual way. That is, if

$$\partial_k\left([\mathfrak{h}_i^{(k)}]\right) = \sum_{j=1}^{p_{(k-1)}} \alpha_{i,j}[\mathfrak{h}_j^{(k-1)}], \alpha_{i,j} \in Z\pi$$

then B_k is the $p_k \times p_{(k-1)}$ matrix

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,p_{(k-1)}} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,p_{(k-1)}} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{p_k,1}, & \alpha_{p_k,2} & \cdots & \alpha_{p_k,p_{k-1}} \end{bmatrix}$$

If we choose a different orientation of core of $\mathfrak{h}_i^{(k)}$, then $[\mathfrak{h}_i^{(k)}]$ is replaced by - $[\mathfrak{h}_i^{(k)}]$ (in the basis of C_k) so that the i^{th} row of B_k is multiplied by -1. If we extend the handle $\mathfrak{h}_i^{(k)}$ along a path representing $\alpha \in \pi$, then $[\mathfrak{h}_i^{(k)}]$ is replaced by $\alpha[\mathfrak{h}_i^{(k)}]$, so that the i^{th} row of B_k is multiplied by α. Thus, by different choices of orientations of cores and paths to the "base point", we can change B_k somewhat. There is another type of modification which we can do on B_k: that is adding a row of B_k to another row of B_k. This is done by using 8.6.3 as follows:

241 Consider two k-handles $\mathfrak{h}_i^{(k)}$ and $\mathfrak{h}_j^{(k)}$ of $\mathscr{H}$, and let $2 \leq k \leq n-2$. We now apply 8.6.3 (Remark 2), with $A_{k-1} = N$, $\overline{\partial(A_{k-1}, X)} = N'$, $\mathfrak{h}_i^{(k)} = \mathfrak{h}_1$, $\mathfrak{h}_j^{(k)} = \mathfrak{h}_2$. This gives a new handle $\mathfrak{h}^{(k)}$, away from $\mathfrak{h}_i^{(k)}$ and $\mathfrak{h}_j^{(k)}$, such that

$$A_{k-1} + \mathfrak{h}^{(k)} + \mathfrak{h}_j^{(k)} \approx A_{k-1} + \mathfrak{h}_i^{(k)} + \mathfrak{h}_j^{(k)}$$

and $\partial[\mathfrak{h}]$, with proper choices, now represents $[\mathfrak{h}_i^{(k)}] \pm \theta[\mathfrak{h}_j^{(k)}]$, (sign prescribed), in $\pi_{k-1}(\partial(A_{k-1}, X))$. Also, we can assume that $\mathfrak{h}^{(k)}$ is away from the attaching tubes of the other handles, so that

$$\begin{aligned} B = (A_{k-1} + \mathfrak{h}^{(k)}) + \mathfrak{h}_j^{(k)} \quad &(\text{other } p_k^{-2} k\text{-handles of } \mathscr{H}) \\ &\overset{\psi}{\approx} A_k, \end{aligned}$$

and ψ can be assumed to be identity on X.

Now (B, X) has an obvious handle presentation $\mathscr{K} = (B_{-1}, \ldots, B_n)$, where

$$\begin{aligned} B_i &= A_{i, i \leq k-1} \\ B_k &= B. \end{aligned}$$

The k^{th} boundary map of $\mathscr{K}$, with the appropriate bases, has a matrix which is the same as B_k except for i^{th} row, which is now replaced by the sum of the i^{th} row + $(\pm\theta)$ times the j^{th} row, corresponding to the relation

$$\partial[\mathfrak{h}^{(k)}] = \partial[\mathfrak{h}_i^{(k)}] \pm \theta\partial[\mathfrak{h}_j^{(k)}]$$

We pull $\mathscr{K}$ to a handle presentation $\mathscr{K}'$ of (A_k, X) by ψ. In $\mathscr{K}$ and
242 $\mathscr{K}'$, the matrices of the boundary maps are the same if we choose the

corresponding bases. And $\mathscr{K}'$ can be extended to handle presentation $\mathscr{H}'$ of (M, X) by adding the $(\geq k+1)$-handles as they are. By doing a finite number of such changes, we have

8.9.2 (Basis Lemma). $\mathscr{H}$ is a handle presentation satisfying 8.9.1, B_k is the matrix of the k^{th} boundary of map of $\mathscr{H}_1^{(k \leq n-2)}$, with respect to bases corresponding to handles. Given any $p_k \times p_k$ matrix E which is the product of elementary matrices, then $\exists$ a handle presentation $\mathscr{H}'$ of (M, X) satisfying 8.9.1

(1) the number of i-handles in $\mathscr{H}$ is the same as the number of i-handles in $\mathscr{H}'$, for all i, and

(2) the matrix of the k^{th} boundary map of $\mathscr{H}'$ with appropriate bases corresponding to handles is $E \cdot B_k$.

As an application of the "Basis Lemma", we will prove a proposition, usually known as the "Existence Theorem for h-cobordisms". Let M be a PL $(n-1)$-manifold; M compact, with or without boundary. The problem is to produce a PL n-manifold W containing M in its boundary such that (W, M) is a h-cobordism with prescribed torsion.

Proposition 8.9.3. *If the dimension of M is greater than 4, then given any $\tau_0 \in Wh(\pi_1, (M))$, there exists a h-cobordism (W, M) with $\tau(W, M) = \tau_0$.*

Proof. Let $A = (a_{ij})$ be a matrix $(m \times m)$ representing τ_0. Consider $N = M \times I$, identify M with $M \times 0$. To (N, M) attach m cancelling pairs of 2- and 3-handles and m 3-handles away from these. Let W' be the resulting manifold: and let $\mathscr{H}$ be the obvious handle presentation of **243**
(W', M); $\mathscr{H}$ satisfies 8.9.1. Then the matrix of the 3^{rd} boundary map of $\mathscr{H}$ with appropriate bases is $\begin{bmatrix} I_m \\ 0_m \end{bmatrix}$. Consider the matrix $\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}$; this is a product of elementary matrices. Hence by 8.9.2, we can obtain a new handle presentation $\mathscr{H}'$ of (W', M) satisfying 8.9.1, such that the number of handles of each index is the same in $\mathscr{H}$ and $\mathscr{H}'$, and the 3^{rd} boundary map of $\mathscr{H}'$ with bases corresponding to handles is

$$\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} I_m \\ 0_m \end{bmatrix} = \begin{bmatrix} A \\ 0_m \end{bmatrix}$$

Thus, if $\mathfrak{R}_1, \ldots, \mathfrak{R}_{2m}$ are the 3-handels and $\mathfrak{h}_1, \ldots, \mathfrak{h}_m$ the 2-handles of $\mathscr{H}'$, and $[\mathfrak{R}_i]$, $[\mathfrak{h}_i]$ denote the corresponding basis elements, then

$$\begin{aligned} \partial[\mathfrak{R}_i] &= \sum_{j=1}^{m} a_{ij}[\mathfrak{h}_j], \quad \text{if} \quad i \leq m \\ \partial[\mathfrak{R}_i] &= 0 \qquad \text{if} \quad i > m. \end{aligned}$$

Let W be the manifold $W' - (\mathfrak{R}_{m+1} \cup \ldots \cup \mathfrak{R}_{2m})$. Let $\mathscr{K}$ be the handle presentation of (W, N) given by $\mathfrak{h}_i$'s and $\mathfrak{R}_i$'s for $i \leq m$. Then the 3[rd] boundary map of $\mathscr{K}$ has the matrix A. Clearly $M \hookrightarrow W$ is a homotopy equivalence (A is non-singular). Since, dually we are attaching $n-2$ and $n-3$ handles to $\overline{\partial(W, M)}$ to get W, and $n - 3 \geq 3$, $\pi_1(\partial(W, M)) \to \pi_1(W)$ is an isomorphism. Hence (W, M) is a h-cobordism with the prescribed torsion τ_0. ◻

244 Again, consider a handle presentation $\mathscr{H}$ satisfying 8.9.1. For $k \leq n-3$, A_k is, upto homotopy obtained by attaching cells of dimension ≥ 3 to $\partial(A_k, X)$. This shows, $\pi_1(\partial(A_k, X)) \to \pi_1(A_k)$ is an isomorphism for $k \leq n - 3$; and hence $\partial\widetilde{(A_k, X)} = \partial(\tilde{A}_k, \tilde{X})$.

We are interested in the following question:

Suppose a k-sphere $\sum \subset (A_k, X)$ represents in $\pi_k(A_k, A_{k-1})$ the element $[\mathfrak{h}]$ corresponding to a particular k-handle. Then, is there a map $f : \sum \to \partial(A_k, X)$ homotopic to the inclusion of $\sum$ in $\partial(A_k, X)$, such that $f|a$ hemisphere of $\sum$ is an embedding onto a core of $\mathfrak{h}$?

We note that $\partial(A_k, X) \cap \partial(A_{k-1}, X)$

$$\begin{aligned} &= \partial(A_k, X) \text{ - (transverse tubes of } k\text{-handles)} \\ &= \partial(A_{k-1}, X) \text{ - (attaching tubes of } k\text{-handles)} \end{aligned}$$

and so $\partial(A_k, X) \cap \partial(A_{k-1}, X)$ will have fundamental group π if either $(n - k - 1) \leq (n - 1) - 3$ or $(k - 1) \leq (n - 1) - 3$, so that $k \leq (n - 3)$ is sufficient. This implies

$$\begin{aligned} \partial(A_k, X)\widetilde{\cap \partial}(A_{k-1}, X) &= \partial\widetilde{(A_k, X)} \cap \partial\widetilde{(A_{k-1}, X)} \\ &= \partial(\tilde{A}_k, \tilde{X}) \cap \partial(\tilde{A}_{k-1}, \tilde{X}). \end{aligned}$$

245 Consider the following diagram:

$$\begin{tikzcd}
\pi_k(\partial(A_k, X), \partial(A_k, X) \cap \partial(A_{k-1}, X)) \arrow[r, "i_1"] & \pi_k(A_k, A_{k-1}) \\
\pi_k(\partial(\tilde{A}_k, \tilde{X}), \partial(\tilde{A}_k, \tilde{X}) \cap \partial(\tilde{A}_{k-1}, \tilde{X})) \arrow[r, "i_2"] \arrow[u, "\alpha_1"'] \arrow[d, "\mathfrak{h}_1"] & \pi_k(\tilde{A}_k, \tilde{A}_{k-1}) \arrow[u, "\alpha_2"'] \arrow[d, "\mathfrak{h}_2"] \\
H_k(\partial(\tilde{A}_k, \tilde{X}), \partial(\tilde{A}_k, \tilde{X}) \cap \partial(\tilde{A}_{k-1}, \tilde{X})) \arrow[r, "i_3"] & H_k(\tilde{A}_k, \tilde{A}_{k-1}).
\end{tikzcd}$$

Here h_1, h_2 are hurewicz maps, i_1, i_2, i_3 are induced by inclusion maps, α_1 and α_2 are the maps induces by $\tilde{M} \to M$. All the induces occuring are ≥ 2. α_1 and α_2 are well known to be isomorphims. h_1 and h_2 are isomorphisms since the pairs are $(k-1)$-connected. By excision, i_3 is an isomorphism. Hence i_2 and i_1 are also isomorphisms. Therefore, a boundary core of $\mathfrak{h}$ and $\sum$ represent the same element in $\pi_k(\partial(A_k, X), \partial(A_k, X) \cap \partial(A_{k-1}, X))$.

Thus the answer to out question is Yes:

8.9.4 Let $\mathscr{H}$ be a handle presentation satisfying the hypothesis 8.9.1. Let k be an integer $\leq n-3$ [or $k \geq 3$, $\pi_1(\partial(A_k, X)) \to \pi_1(A_k)$ is an isomorphism, $k \leq n-1$]. Then two geometric objects, representing the same element of $\pi_k(A_k, A_{k-1})$, also represent the same element of $\pi_k(\partial(A_k, X), \partial(A_k, X) \cap \partial(A_{k-1}, X))$. In particular, if $\sum \subset \partial(A_k, X)$ is a k-sphere, representing the element $[\mathfrak{h}]$ in $\pi_k(A_k, A_{k-1})$; then it represents **246**
the element $[\mathfrak{h}]$ in $\pi_k(\partial(A_k, X), \partial(A_k, X) \cap \partial(A_{k-1}, X))$. This means that there is a homotopy in $\partial(A_k, X)$ from the identity map of $\sum$ of a map taking the upper hemisphere of $\sum$ in a $1-1$ way onto a (boudnary) core of $\mathfrak{h}$, and taking the lower hemisphere of into $\partial(A_k, X) \cap \partial(A_{k-1}, X)$; in particular the end result of $\sum$ will not intersect any other handles.

If $\sum$ is the attaching sphere of a $(k+1)$-handle $\mathfrak{K}$ and $\partial_{k+1}([\mathfrak{K}]) = [\mathfrak{h}]$, we have the above situation. We would like to get a suitable isotopy from the above homotopy information, to cancel the handles corresponding $\mathfrak{K}$ and $\mathfrak{h}$ in some $\mathscr{H}_f$. This is provided by the following lemma. Since the proof of this lemma is rather long and seems to be of some general interest, we will postpone the proof to the last section.

8.9.5 (Isotopy Lemma). With the hypotheses of 8.9.4, if n addition, $n \geq 6$ and $k \leq n-4$, then there is an isotopy in $\partial(A_k, X)$ carrying $\sum$ to another k-sphere $\sum'$, such that $\sum'$ intersects a transverse sphere of $\mathfrak{h}$ in one point transversally and does not intersect the other k-handles.

8.10 Proofs of Theorems A and B

In this section, we will prove Theorems A and B assuming the Isotopy Lemma, which will be proved in the next section.

First let us see what are the types of manifolds and presentations that we have to consider. Theorem A, for $\ell \leq 1$ is proved in 8.7. So, we can assume $\ell \geq 2$, and hence $n \geq 6$. For Theorem B, $\ell = n$ and
247 $n \geq 6$, by hypothesis. So again using 8.7, it is enough to consider handle presentations satisfying 8.9.1, and in addition $n \geq 6$.

We start with two observations concerning the matrices of the boundary maps:

8.10.1 Let $\mathscr{H}$ be handle presentation satisfying 8.9.1, $\mathfrak{h}_1^{(i)}, \ldots, \mathfrak{h}_{p_i}^{(i)}$ be the i-handles of $\mathscr{H}$. Let $B_{k+1} = (\alpha_{ij})$ be the matrix of the $(k+1)^{\text{st}}$ boundary map ∂_{k+1} with respect to preferred bases. That is,

$$\partial_{k+1}\left(\left[\mathfrak{h}_i^{(k+1)}\right]\right) = \sum_{j=1}^{p_k} \alpha_{ij}[\mathfrak{h}_j^{(k)}], \quad \alpha_{ij} \in Z(\pi).$$

Suppose $\mathfrak{h}_1^{(k)}$ and $\mathfrak{h}_1^{(k+1)}$ can be cancelled. Then we have formed a handle presentation $\mathscr{H} - (\mathfrak{h}_1^{(k)}, \mathfrak{h}_1^{(k+1)}) = (B_{-1}, \ldots, B_n)$ say, of (M, X) as follows:

$$\begin{aligned}
&B_i = f(A_i) \quad \text{for} \quad i < k\\
&B_k = f(A_k - \mathfrak{h}_1^{(k)}) = A_k + \mathfrak{h}_1^{(k+1)}\\
&B_i = A_i \quad \text{for} \quad i > k.
\end{aligned}$$

Here f is an equivalence $A_k - \mathfrak{h}_1^{(k)} \approx A_k + \mathfrak{h}_1^{(k+1)}$ mapping X onto itself. If the attaching tube of $\mathfrak{h}_1^{(k+1)}$ does not intersect any other k-handle except $\mathfrak{h}_1^{(k)}$, f can be assumed to be identity on all $\mathfrak{h}_i^{(k)}$, $i \geq 2$.

We assume that this is the case. Now $\mathfrak{h}_2^{(k)}, \dots, \mathfrak{h}_{p_k}^{(k)}$ are all the k-handles and $\mathfrak{h}_2^{(k+1)}, \dots, \mathfrak{h}_{p_{(k+1)}}^{(k+1)}$ are all the $(k+1)$-handles of $\mathscr{H} - (\mathfrak{h}_1^{(k)}, \mathfrak{h}_1^{(k+1)})$. Thus (by abuse of notation) $[\mathfrak{h}_2^{(k)}], \dots, [\mathfrak{h}_{p_k}^{(k)}]$ is a basis of $\pi_k(B_k, B_{k-1})$ and $[\mathfrak{h}_2^{(k+1)}], \dots, [\mathfrak{h}_{p_{(k+1)}}^{(k+1)}]$ is a basis of $\pi_{k+1}(B_{k+1}, B_k)$. Let ∂'_{k+1} denote the **248**
$(k+1)^{\text{st}}$ boundary map of $\mathscr{H} - (\mathfrak{h}_1^{(k)}, \mathfrak{h}_1^{(k+1)})$. Consider the following commutative diagram ($A_{k+1} = B_{k+1}, A_k \subset B_k, A_{k-1} \subset B_{k-1}$):

$$\begin{tikzcd}
\pi_{k+1}(A_{k+1}, A_k) \arrow[r, "\partial"] \arrow[d, "i_{1*}"] & \pi_k(A_k) \arrow[r, "j"] \arrow[d, "i_{2*}"] & \pi_k(A_k, A_{k-1}) \arrow[d, "i_{3*}"] \\
\pi_{k+1}(B_{k+1}, B_k) \arrow[r, "\partial'"] & \pi_k(B_k) \arrow[r, "j'"] & \pi_k(B_k, B_{k-1})
\end{tikzcd}$$

In this diagram, the vertical maps are induced by inclusion, the horizontal maps are canonical maps, and $j \circ \partial = \partial_{k+1}$, $j' \circ \partial' = \partial'_{k+1}$. Now

$$\begin{aligned}
&i_{1_*}\left([\mathfrak{h}_1^{(k+1)}]\right) = 0 \\
&i_{1_*}\left([\mathfrak{h}_i^{(k+1)}]\right) = [\mathfrak{h}_i^{(k+1)}] \quad \text{for} \quad i \geq 2.
\end{aligned}$$

and

$$\begin{aligned}
&i_{3_0}\left([\mathfrak{h}_1^{(k)}]\right) = 0, \\
&i_{3_*}\left([\mathfrak{h}_i^{(k)}]\right) = \left([\mathfrak{h}_i^{(k)}]\right) \quad \text{for} \quad i \geq 2.
\end{aligned}$$

Hence, for $i \geq 2$,

$$\begin{aligned}
&\partial'_{k+1}\left([\mathfrak{h}_i^{(k+1)}]\right) \\
&= \partial'_{k+1} \circ i_{1_*}\left([\mathfrak{h}_i^{(k+1)}]\right) \\
&= j' \circ \partial' \circ i_{1_*}\left([\mathfrak{h}_i^{(k+1)}]\right) \\
&= i_{3_*} \circ j \circ \partial\left([\mathfrak{h}_i^{(k+1)}]\right) \\
&= i_{3_*} \circ \partial_{k+1}\left([\mathfrak{h}_i^{(k+1)}]\right) \\
&= i_{3_*}\left(\sum_{i=1}^{p_k} \alpha_{ij}[\mathfrak{h}_j^{(k)}]\right)
\end{aligned}$$

$$= \sum_{i=2}^{p_k} \alpha_{ij}\left([\mathfrak{h}_j^{(k)}]\right)$$

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Thus, if the matrix of ∂_{k+1} is (α_{ij}), then the matrix of ∂'_{k+1} is (α_{ij}), $i \geq 2$, $j \geq 2$. This we have as long as the attaching tube of $\mathfrak{h}_1^{(k+1)}$ keeps away from the transverse tubes of the handles $\mathfrak{h}_i^{(k)}$, $i \geq 2$. (It is easy to see that $\alpha_{1,2} = \ldots = \alpha_{1,pk} = 0$, in this case). It does not matter even if the attaching tubes of other $(k+1)$-handles intersect the transverse tube of $\mathfrak{h}_1^{(k)}$.

8.10.2 If $f : A_k \to A_k$ is an equivalence isotopic to the identity leaving X fixed, then in $\mathscr{H}^f$ and $\mathscr{H}$, the attaching spheres of the corresponding $(k+1)$-handles represent the same elements in $\pi_k(A_k)$. Since the $(k+1)^{\text{st}}$ boundary maps are factored through $\pi_k(A_k)$, the corresponding matrices are the same after the choice of obvious bases in $\mathscr{H}$ and $\mathscr{H}^f$, and hence in $\mathscr{H}$ and $\mathscr{H}(f^{-1})$.

Proof of Theorem A.

Step 1. Let $\mathscr{H} = (A_{-1}, \ldots, A_n)$ be a handle presentation of (M, X) satisfying 8.9.1. We are given that (M, X) is ℓ-connected, then we know that the sequence

$$\pi_{\ell+1}(A_{\ell+1}, A_\ell) \to \pi_\ell(A_\ell, A_{\ell-1}) \to \ldots \pi_2(A_2, A_{-1}) \to 0$$

250 is exact.

Suppose that we have already eliminated upto handles of index $(i-1)$, that is in $\mathscr{H}$, $A_{-1} = A_0 = \ldots = A_{i-1}$; then by (*), $\pi_{i+1}(A_{i+1}, A_i) \xrightarrow{\partial_{i+1}} \pi_i(A_i, A_{i-1})$ is onto. Let B_{i+1} be the matrix $(p_{i+1} \times p_i)$ of π_{i+1} with bases corresponding to handles. Then, there exists a $(p_{i+1} + p_i) \times (p_{i+1} + p_i)$ matrix E, which is the product of elementary matrices, such that

$$E \times \begin{bmatrix} B_{i+1} \\ 0_{p_i} \end{bmatrix} = \begin{bmatrix} I_{p_i} \\ 0_{p_{i+1}, p_i} \end{bmatrix}.$$

If we attach p_i cancelling pairs of $(i+1)$, $(i+2)$-handles away from the handles of index $\leq (i+2)$ to the i^{th} level of $\mathscr{H}$, then in the resulting handle presentation $(B_{-1},\dots,B_n)$ of (M,X), the matrix of $\pi_{i+1}(B_{i+1},B_i)\to\pi_k(B_i,B_{i-1})$ with appropriate bases is

$$\begin{bmatrix} B_{i+1} \\ 0_{p_i} \end{bmatrix}.$$

Then, by the Basis Lemma, we can obtain a handle presentation of (M,X) satisfying 8.9.1, with exactly the same number of handles of each index as above, but $(i+1)^{\text{st}}$ boundary matrix will now be

$$E\times\begin{bmatrix} B_{i+1} \\ 0_{p_i} \end{bmatrix}=\begin{bmatrix} I_{p_i} \\ 0_{p_{i+1},p_i} \end{bmatrix}$$

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This means that starting from $\mathscr{H}$, we can obtain a handle presentation $\mathscr{K}=(C_{-1},\dots,C_n)$ of (M,X) such that

(1) $\mathscr{K}$ satisfies 8.9.1, and there are no handles of indices $\leq i-1$

(2) the $(i+1)^{\text{st}}$ boundary map of $\mathscr{K}$ has the matrix $\left[\begin{smallmatrix} I_{p_i} \\ 0 \end{smallmatrix}\right]$.

Now we can eliminate the i-handles one at a time as follows:

Step 2. Consider $\mathfrak{h}_1^{(i+1)}$ and $\mathfrak{h}_1^{(i)}\cdot\partial_{i+1}([\mathfrak{h}_1^{(i+1)}])=[\mathfrak{h}_1^{(i)}]$; and $i\leq n-4$. Hence by the Isotopy Lemma, there is an equivalence f of $\partial(C_i,X)$, such that f takes an attaching sphere S_1 of $\mathfrak{h}_1^{(i+1)}$ to another i-sphere S_1' and S_1' intersects a transverse sphere of $\mathfrak{h}_1^{(i)}$ at one point transversally. Moreover it can be assumed that f (attaching tube of $\mathfrak{h}_1^{(i+1)}$) does not intersect the transverse tubes of the other i-handles. f can be extended to an equivalence f of C_i taking X onto itself and in $\mathfrak{K}^f$ the handles corresponding $\mathfrak{h}_1^{(i)}$ and $\mathfrak{h}_1^{(i+1)}$ can be nearly cancelled. By 8.5.8, there is an equivalence g of C_i, so that in $(\mathscr{K}^f)^g=\mathscr{K}^{(gf)}$, the handles corresponding $\mathfrak{h}_1^{(i)}$ and $\mathfrak{h}_1^{(i+1)}$ can be cancelled. Again, we can require that $g\circ f$ (attaching tube of $\mathfrak{h}_1^{(i+1)}$) should not intersect the transverse tubes of handles $\mathfrak{h}_j^{(i)}$, $j\geq 2$. Consider $\mathscr{K}_{(gf)^{-1}}$. This is a handle-presentation of (M,X), and in $\mathscr{K}_{(gf)^{-1}}$ 252
the handles $\mathfrak{K}_1^{(i)}$ and $\mathfrak{K}_1^{(i+1)}$ say, corresponding to $\mathfrak{h}_1^{(i)}$ and $\mathfrak{h}_1^{(i+1)}$ can be cancelled. By 8.10.1, and since f and g can be assumed to be isotopic to identity, the $(i+1)^{\text{st}}$ boundary map of $\mathscr{K}_{(gf)^{-1}}-(\mathfrak{K}_1^{(i)},\mathfrak{K}_1^{(i+1)})$ has the

matrix $\begin{bmatrix} I_{(p_i-1)} \\ 0 \end{bmatrix}$. Hence, we can go on repeating step 2 to obtain a handle presentation of (M, X) without handles upto index i.

Thus, inductively, the first part of Theorem A is proved. The second part is clear.

Proof of Theorem B: By Theorem A, we can assume that there is a handle presentation $\mathscr{H}$ of (M, X) with handles of indices $(n-3)$ and $(n-4)$ only. $\mathscr{H}$ obviously satisfies 8.9.1. Consider the map

$$\partial_{n-3} : \pi_{n-3}(A_{n-3}, A_{n-4}) \to \pi_{n-4}(A_{n-4}, A_{n-5}).$$

Here $A_{-1} = \ldots = A_{n-5}$. Let A be the matrix of ∂_{n-3} with respect to bases corresponding to handles. A is a nonsingular matrix, say $m \times m$ matrix. Since $\tau(M, X) = 0$, A represents the 0-element in $Wh(\pi)$. Hence for some $q \leq m$ there exists an $(m+q) \times (m+q)$ matrix E which is the product of elementary matrices, such that

$$E \times \begin{bmatrix} A & 0 \\ 0 & I_q \end{bmatrix} = I_{m+q}$$

Now we add q cancelling pairs of $(n-3)$- and $(n-4)$-handles to A_{n-5} away from the other handles, so that in the new handle presentation, say $\mathscr{K}$, of (M, X), the $(n-3)^{\text{rd}}$ boundary map of $\mathscr{K}$ has the matrix

$$\begin{bmatrix} A & 0 \\ 0 & I_q \end{bmatrix}.$$

253 Then by the Basis Lemma, we can obtain a new handle presentation $\mathscr{K}'$ of (M, X) with exactly $(m+q)$ handles of indices $(n-3)$ and $(n-4)$ and no others, and such that the matrix of the $(n-3)^{\text{rd}}$ boundary map of $\mathscr{K}'$ with respect to bases corresponding to handles is I_{m+q}. Now, by a repeated application of Step 2 in the proof the Theorem A, all the handles can be eliminated so that $M \searrow X$.

8.11 Proof of the Isotopy Lemma

We begin with some elementary lemmas.

Lemma 8.11.1. *Let $Q \subset P$, $Y \times \Delta \subset X$ be polyhedra, where Δ is k-cell and* $\dim Q \leq k$ *and $f : p \to X$ a polyhedral map. Then the set of points $\alpha \in \Delta$ such that $Q \cap f^{-1}(Y \times \alpha) = \emptyset$ contains an open and dense subset of Δ.*

Proof. Let $Q' = f(Q) \cap (Y \times \Delta)$. Then $\dim Q' \leq \dim Q < k$. The projection of Q' to Δ does not cover most of the points of k-dimensional Δ. Any point α not belonging to the projection of Q' to Δ will satisfy $Q \cap f^{-1}(Y \times \alpha) = \emptyset$. □

Lemma 8.11.2. *If* $\dim Q = k$ *in the above, then the set of points $\alpha \in \Delta$ such that $Q \cap f^{-1}(Y \times \alpha)$ is* 0*-dimensional contains an open and dense subset of Δ.*

Proof. Triangulate the projection of Q' to Δ. If $\mathscr{A}$ is the simplicial presentation of Δ with respect to which this map is simplicial, then every point α of $\Delta - |\mathscr{A}_{k-1}|$ will have the above property by 4.2. □

Let $f : P \to X$ be a nondegenerate map, simplicial with respect to the presentations $\mathscr{P}$, $\mathscr{X}$ of P, X. Let $\sum(f)$ denote the closure of the set $S(f) = \{x \in P | \exists y \in P, y \neq x, f(y) = f(x)\}$ (see 5.4). $\sum(f)$ is covered by **254**
a subpresentation of $\mathscr{P}$, call it $\sum$.

8.11.3 If σ is a principal simplex of $\sum$, then $f||St(\sigma, \mathscr{P})|$ is an embedding.

Proof. Since f is nondegenerate it maps $Lk(\sigma, \mathscr{P})$ into $Lk(f\sigma, \mathscr{X})$ and on $|St(\sigma, \mathscr{P})|$ it is the join of $\overline{\sigma} \to \overline{f\sigma}$ and $|Lk(\sigma, \mathscr{P}| \to |Lk(f\sigma, \mathscr{X})|$. The map $|Lk(\sigma, \mathscr{P})| \to |Lk(f\sigma, \mathscr{X})|$ is an embedding; otherwise if $\tau_1 \neq \tau_2$, τ_1, $\tau_2 \in Lk(\sigma, \mathscr{P})$ and $f\sigma_1 = f\sigma_2$, then $f(\sigma\tau_1) = f(\sigma\tau_2)$ so that $\sigma\tau_1 \in \sum$, contrary to the assumption that σ is a principal simplex of $\sum$. Hence the map $|St(\sigma, \mathscr{P})| \to |St(f(\sigma), \mathscr{X})|$ being the join of embeddings is an embedding. □

Proof of the Isotopy Lemma for $k \leq n - 5$: The situation is: We have a handle presentation $\mathscr{H}$ of a relative n-manifold (M, X) which is a special case, and $\mathscr{H}$ satisfies the hypothesis 8.9.1. We have k-sphere S

(what was called $\sum$ in 8.9.4 and 8.9.5) in $\partial(A_k, X)$ representing $[\mathfrak{h}]$ in $\pi_k(A_k, A_{k-1})$ where $\mathfrak{h}$ is a k-handle. We deduced in 8.9.4 that in this case if $k \leq n-3$ there is a homotopy $h : S \times I \to \partial(A_k, \mathscr{X})$ such that $h_0 =$ embedding $S \subset \partial(A_k, X)$ and h_1^{-1} (transverse tubes of all k-handles) is a k-cell C which is mapped by h_1 isomorphically onto a core of $\mathfrak{h}$, so that $h_1(S - C) \subset \partial(A_k, X) \cap \partial(A_{k-1}, X)$.

In the isotopy Lemma, we have further assumed that $k \leq n-4$. We first prove the simpler case when $k \leq n-5$, that is when the co-dimension of S in $\partial(A_k, X)$ is ≥ 4.

255 We can by general position suppose $\sum(h)$ has dimension $\leq 2(k+1)-(n-1) = 2k+3-n$.

Now $\mathfrak{h}$ is polyhedrally equivalent to $D^k \times \Delta^{n-k}$, with the transverse tube of $\mathfrak{h}$ corresponding to $D \times \partial\Delta \subset (A_k, X)$. For any point $\alpha \in \operatorname{int} D$, $\alpha \times \partial\Delta$ is a transverse sphere; and any such transverse sphere will intersect the core $h_1(C)$ transversally in exactly one point, since $h_1(C)$ corresponds to $D \times \beta$, for some $\beta \in \partial\Delta$.

We try to apply Lemma 8.11.1 to this situation. Define

$$Q = \text{“Shadow”} \sum(h) = [\mathrm{Proj}_S \sum(h)] \times I$$

$$P = S \times I$$

$$P \xrightarrow{f} X \supset Y \times \Delta^k \text{ becomes}$$

$$S \times I \xrightarrow{h} \partial(A_k, X) \supset \text{ transverse tube of } \mathfrak{h} \approx \partial\Delta \times D^k.$$

The crucial hypothesis now is $\dim Q < k$. Since, in general $\dim(\mathrm{proj}_S \sum(h)) \leq \dim \sum(h)$, we have $\dim Q \leq \dim \sum(h) + 1 \leq 2k+4-n$. To have this $< k$ is exactly where we need $k \leq n-5$.

The conclusion then is:

$\alpha \times \partial\Delta$. There exists a transverse sphere T of $\mathfrak{h}$ of the form $\alpha \times \partial\Delta$, for some $\alpha \in \operatorname{Int} D$, so that $h^{-1}(T)$ does not intersect the shadow of the singularities $\sum(h)$ or, what amounts to the same, the “shadow” of $h^{-1}(T)$, namely

$$Z = [\mathrm{proj}_S\, h^{-1}(T)] \times I \subset S \times I$$

does not intersect $\sum(h)$. Hence there is some regular neighbourhood N of Z in $S \times I$, with $N \cap \sum(h) = \emptyset$. This implies, since h_0 is an embedding, that $h|S \times 0 \cup N$ is an embedding.

We clearly have $N \searrow N \cap S \times 0$, and these are $(k+1)$- and k- **256**
manifolds, $N \cap S \times 0 \subset \partial N$. Thus $h(S \times 0) = S$ and $h[S \times 0 - (N \cap S \times 0) + (\partial N - S \times 0)] = S'$ differ in $\partial(A_k, X)$ by cellular moves along the manifold $h(N)$. Therefore (by 7.1.8) there is an isotopy of $\partial(A_k, X)$ taking S onto S'. By construction all of $h^{-1}(T)$ is in N, and S' contains only $h(\overline{\partial N - S \times 0})$ in $h(N)$, and this will intersect T at $h(h^{-1}(T) \cap S \times 1)$, that is at point (corresponding to $\alpha \times \beta$) transversally.

By being only a bit more careful, considering the transverse tubes of other k-handles, we can arrange for S' not to intersect the other k-handles at all (if T' is a transverse sphere of some k-handle other than $\mathfrak{h}$, then $h^{-1}(T') \cap S \times 1 = \emptyset$, and there is an isotopy of $\partial(A_k, X)$ carrying $\partial(A_k, X)$-small regular neighbourhoods of prescribed transverse sphere of the other k-handles ot $\partial(A_k, X)$-transverse tubes of the other k-handles).

Remark. This already gives Theorem C for $n \geq 8$.

The case $k = n - 4$.

In case $k = n-4$, $n \geq 6$, the above result is still true, but this involves some delicate points.

Since $n \geq 6$, we have (for $k = n-4$) the crucial number $2k+3-n > 0$.

We consider, as before $h : S \times I \to \partial(A_k, X)$ in general position, so that $\dim \sum(h) \leq 2k + 3 - n$. Remembering $S = h(S \times 0)$, we further use general position so that $h^{-1}(S) \cap S \times (0, 1]$ is of dimension $\leq k + (k + 1) - (n - 1) = 2k + 2 - n$, and call

$$\theta(h) = \text{ closure } (h^{-1}(S) \cap S(0, 1]).$$

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Make h simplicial, say with reference to $\mathscr{S}$ of $S \times I$, and refine $\mathscr{S}$ to $\mathscr{S}'$ so that $\theta(h)$ is covered by a subpresentation θ, $\sum(h)$ by a subpresentation $\sum$, and the projection $S \times I \to S$ is simplicial on $\mathscr{S}'$.

Now we have to pick our transverse sphere $T = \alpha \times \partial\Delta^{n-k}$ in the transverse tube $D^k \times \partial D^{n-k}$ so that

(1) $h^{-1}(T)\cap$ shadow $\sum(h)$ is 0-dimensional

(2) $h^{-1}(T)\cap\sum(h)=\emptyset$

(3) $h^{-1}(T)\cap$ shadow $\{(2k+2-n)$-skeleton of $\sum\}=\emptyset$

On $S\times 0\cup$ a neighbourhood of [shadow $h^{-1}(T)$], h is a local embedding, using Lemma 8.11.3.

Let $Q=$ shadow $h^{-1}(T)$. The finite set of points $Q\cap\sum(h)$ does not intersect any point of $h^{-1}(T)$. Each point say $x\in Q\cap\sum(h)$ belongs to a $(2k+3-n)$-simplex of $\sum_j$ say σ_x. Since σ_x has dimension ≥ 1, we can mover $S\times I$ in a tiny neighbourhood of x by a polyhedral equivalence $f: S\times I\to S\times I$ so as to move x around on σ_x, that is, so that

$$f(Q)\cap\sum(h)=Q\cap\sum(h)-\{x\}+\{x'\},$$

where the choice of x' ranges over an infinite set. f will not move $h^{-1}(T)$ nor will it move $S\times 0$. There are only a finitely many points to worry about, and so we can find a polyhedral equivalence $f: S\times I\to S\times I$, leaving $h^{-1}(T)\cup S\times 0$ fixed, such that the set of points $f(Q)\cap\sum(h)$ are mapped by h into pairwise distinct points.

258 At this moment, we see that on $S\times 0\cup f(Q)$, h is an embedding. Since h is a local embedding on some neighbourhood of $f(Q)$ (we restrict f close to the identity so that $f(Q)\subset S\times I$ - $\{(2k+2-n)$-skeleton of $\sum\}$), and an embedding on $f(Q)$; hence it is an embedding on some neighbourhood of $f(Q)$.

$\theta(h)$ is well out of the way, and so h actually embeds all of $S\times 0\cup$ (a neighbourhood of $f(Q)$).

We now proceed as before, using $f(Q)$ to move around along.

This trick looks a bit different from piping, which is that we would have to do in the case $n=5$, $k=1$; this was the case when we had a null homotopic 1-sphere $\subset$ 4-manifold unknotted.